LECTURE II - MATRIX MULTIPLICATION (II) (SECTION 2.3)

Now that we've seen how to multiply matrices, today we'll discuss some neat consequences of matrix multiplication.

I - \( T(x) \)

First of all, matrix multiplication allows us to calculate \( T(x) \) without calculating \( T(x) \) itself (seems paradoxical, but it is true)

**Fact** IF \( T : V \to W \), \( \beta \) basis of \( V \), \( \gamma \) basis of \( W \), THEN

\[
[T(x)]_{\gamma} = [T]_{\beta}^\gamma [x]_\beta \quad \text{(For all } x \in V) \]

VECTOR \hspace{1cm} MATRIX \hspace{1cm} VECTOR

(To calculate coords of \( T(x) \), just need to calculate coords of \( x \) and multiply this by the matrix of \( T \))

**Why?** LET \( \beta = \{ v_1, \ldots, v_n \} \), \( x \in V \)

THEN \( x = x_1 v_1 + \cdots + x_n v_n \) FOR SOME \( x_1, \ldots, x_n \in \mathbb{F} \)

**Note** \( [x]_{\beta} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \)

THEN \( [T(x)]_{\gamma} = [T(x_1 v_1 + \cdots + x_n v_n)]_{\gamma} \)

\[
= [x_1 T(v_1) + \cdots + x_n T(v_n)]_{\gamma} \\
= \begin{bmatrix} T(v_1) \\ \vdots \\ T(v_n) \end{bmatrix}_{\gamma} [x_1 \ldots x_n]_{\gamma}^\top \quad \text{Last Time} \]

\[
= \begin{bmatrix} T(v_1) \\ \vdots \\ T(v_n) \end{bmatrix}_{\gamma} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}_{\gamma} \]

\]
\[ \text{EX} \quad T : P_3 \rightarrow P_2 \quad p = \{1, x, x^2, x^3\} \quad y = \{1, x, x^2\} \]

\[ T(p) = p' \]

\[ \text{CALCULATE} \quad T(2 + 4x + 3x^2 + 5x^3) \quad (\text{W/O CALCULATING ANY DERIVATIVE}) \]

\[ p = \frac{2}{3} + 4x + 3x^2 + 5x^3 \]

\[ [p]_p = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} \]

Then \[ [T(p)]_y = [T]_p [p]_p \]

\[ = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix} \]

Last week

\[ [T(p)]_y = \begin{bmatrix} 4 \\ 6 \\ 15 \end{bmatrix} \]

so \[ T(p) = 4(1) + 6(x) + 15(x^2) = 4 + 6x + 15x^2 \]

(THUS IT PROBABLY HOW YOUR CALCULATION CALCULATED DERIVATIVE.)

(Notice how we took this abstract problem and turned it into a concrete problem about polynomials, you'll see this a lot in lab.)
II. $\text{LA}$

So far, given $A \in LT$, we were able to extract a matrix $A$ out of it, namely $A = (T)^{\otimes y}$.

What about the converse? Given a matrix $A$, is it possible to obtain a $LT$ $T$?

Yes, indeed! In general, they call it $SF$, but here we call it $LA$:

**Def**: If $A$ is a $m \times n$ matrix, then $LA : F^n \rightarrow F^m$ is defined by:

$$LA(x) = Ax$$

**Ex**: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $LA : F^2 \rightarrow F^3$

$$LA(x, y, z) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x + 2y + 3z, 4x + 5y + 6z)$$

*Do not confuse $A$ (matrix) with $LA$ (LT)!!*

That said, even though they're not the same, we'll see that they share similar properties.
PROPERTIES: \( A: M \times N, \quad L_A: F^N \rightarrow F^M, \quad L_A(x) = Ax \)

\((e_1, \ldots, e_N) = e = \text{standard basis of } F^N\) \((1,0,0), (0,1,0), \text{ etc.}\)

\(y = \text{standard basis of } F^M\)

(a) \(L_A: F^N \rightarrow F^M\) is linear

\(L_A(x + cy) = A(x + cy) = Ax + cAy = L_A(x) + cL_A(y)\)

\(\text{Note if } y \in F^M \text{ and } y = \text{standard basis, then } \{y\}_y = x\)

\([Ay]_y = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\)

(b) \([L_A]_y = A\)

\(\text{Note}\)

\(j^{th}\) column of \(A = A e_j = \begin{bmatrix} A(e_j) \\ \in F^M \end{bmatrix} = \begin{bmatrix} [L_A(e_j)]_y \\ y \in F^N \end{bmatrix} = j^{th}\) column of \([L_A]_y\)

(c) \(L_A = L_B \iff A = B\)

\(\text{Why?}\)

\(L_A = L_B \iff [L_A]_y = [L_B]_y \iff A = B\)

(d) \(L_{A+B} = L_A + L_B\)

\(\text{Why?}\)

\(L_{A+B}(x) = (A + B)x = Ax + Bx = L_A(x) + L_B(x)\)

(e) \(\text{If } T: F^N \rightarrow F^M, \text{ then } T = L_c \text{ for some (unique) } c \in F^N\)

\(\text{Why?}\)

\(\text{Let } c = [T]_y, \text{ then}\)

\([L_c(x)]_y = L_c(x) = Cx = [T]_y [x]_y = [T(x)]_y\)

\(\text{Uniqueness: } \text{if } T = L_c \text{ and } T = L_d, \text{ then } L_c = L_d \iff c = d\)
\[(4) \quad L_{AB} = L_A (L_B)\]
\[(4) \quad L_{AB} (e_j) = (AB) (e_j) = A (B e_j) = A (L_B (e_j)) = L_A (L_B (e_j)) = (L_A L_B) (e_j)\]

Since \( \{e_1, \ldots, e_n\} \) is a basis of \( \mathbb{F}^n \), get \( L_{AB} = L_A L_B \)

Page of (4) (Not obvious if we haven't shown associativity of matrix multiplication!)

Suppose \( B = \begin{bmatrix} U_1 & \cdots & U_p \end{bmatrix} \)

Then \( (AB) (e_j) = \left( A \begin{bmatrix} U_1 & \cdots & U_p \end{bmatrix} \right) (e_j) \)

\[= \begin{bmatrix} AU_1 & \cdots & AU_p \end{bmatrix} (e_j) \]

\[= j\textsuperscript{th} \text{ col of} \begin{bmatrix} AU_1 & \cdots & AU_p \end{bmatrix} \]

\[= AU_j \]

\[= A (e_j) \quad \checkmark\]

\[(9) \quad L_N = n \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix} \]
\[L_{Nw} = L_{\mathbb{F}^w} \quad \begin{bmatrix} x \end{bmatrix}_{\mathbb{F}^w} = x \]

(Want to convince you of is that, in some sense, matrices are like \( LT \), \( "A\) is like \( L_A"\))

\[\text{Matrices} \quad LT \]

(Given \( A \), can define \( L_A \).

Given \( T \), can define \( [T]_A \))
Lastly, here is why $L_A$ is so cool!

**Fact:** $A(BC) = (AB) C$ (associativity)

(\textbf{Bad Way: Use def of matrix multiplication})

\begin{align*}
L_A(BC) &= L_A(L_{BC}) \\
&= L_A(L_B(L_C)) \quad \text{(LT are associative)} \\
&= (L_A L_B) L_C \quad \text{(Use function one!)} \\
&= (L_{AB}) L_C \quad \text{(4)} \\
&= L_{(AB)} C \quad \text{(4)}
\end{align*}

$\Rightarrow L_A(BC) = L_{(AB)} C$

(c) $A(BC) = (AB) C \quad \text{(Wow!)}$