WEDNESDAY, APRIL 24, 2019

LECTURE 11 – MATRIX MULTIPLICATION (II) (SECTION 2.3)

NOW THAT WE'VE SEEN HOW TO MULTIPLY MATRICES, TODAY WE'LL DISCUSS SOME NEAT CONSEQUENCES OF MATRIX MULTIPLICATION:

I – $T(x)$

FIRST OF ALL, MATRIX MULTIPLICATION ALLOWS US TO CALCULATE $T(x)$ WITHOUT CALCULATING $T(x)$!! (SEEMS PARADOXICAL, BUT IT'S TRUE)

FACT IF $T: V \to W$, β BASIS OF V, γ BASIS OF W, THEN

$$\underbrace{[T(x)]_\gamma}_{\text{VECTOR}} = \underbrace{[T]_\beta^\gamma}_{\text{MATRIX}} \underbrace{[x]_\beta}_{\text{VECTOR}} \qquad (\text{FOR ALL } x \in V)$$

(TO CALCULATE COORDS OF $T(x)$, JUST NEED TO CALCULATE COORDS OF x AND MULTIPLY THIS BY THE MATRIX OF T)

WHY? LET $\beta = \{v_1, \dots, v_n\}$, $x \in V$,

THEN $x = x_1 v_1 + \dots + x_n v_n$ FOR SOME $x_1, \dots, x_n \in V$

NOTE $[x]_\beta = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

THEN
$$\begin{aligned}
[T(x)]_\gamma &= [T(x_1 v_1 + \dots + x_n v_n)]_\gamma \\
&= [x_1 T(v_1) + \dots + x_n T(v_n)]_\gamma \\
&= x_1 [T(v_1)]_\gamma + \dots + x_n [T(v_n)]_\gamma \qquad \checkmark\ [\ ]_\gamma \text{ IS LINEAR} \\
&= \begin{bmatrix} [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \checkmark \text{ LAST TIME}
\end{aligned}$$

$$= [T]_\beta^\gamma [x]_\beta \quad (\text{DEF OF } [T]_\beta^\gamma)$$

(LET ME SHOW YOU WHY IT'S SO USEFUL)

EX $\quad T : P_3 \to P_2 \qquad \beta = \{1, x, x^2, x^3\}$

$\qquad\qquad\qquad\qquad\qquad\qquad \gamma = \{1, x, x^2\}$

$$T(p) = p'$$

CALCULATE $T(2+4x+3x^2+5x^3)$ (W/O CALCULATING ANY DERIVATIVES)

$$p = \underline{\underline{2}} + \underline{\underline{4}}x + \underline{\underline{3}}x^2 + \underline{\underline{5}}x^3$$

$$[p]_\beta = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix}$$

THEN $\quad [T(p)]_\gamma = [T]_\beta^\gamma [p]_\beta$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{\text{LAST WEEK}} \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \end{bmatrix}$$

$$[T(p)]_\gamma = \begin{bmatrix} 4 \\ 6 \\ 15 \end{bmatrix}$$

SO $\quad T(p) = \underline{\underline{4}}(1) + \underline{\underline{6}}(x) + \underline{\underline{15}}(x^2) = 4 + 6x + 15x^2$

γ

(THIS IS PROBABLY HOW YOUR CALCULATOR CALCULATES DERIVATIVES)

(NOTICE HOW WE TOOK THIS ABSTRACT PROBLEM AND TURNED IT INTO A CONCRETE PROBLEM ABOUT MATRICES; YOU'LL SEE THIS A LOT IN 121AB)

II – L_A

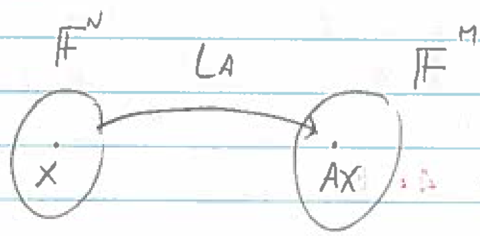
<u>So far</u> <u>given</u> a LT T, we were able to extract a matrix A out of LT, namely $A = [T]_\beta^\gamma$

What about the converse? <u>Given</u> a matrix A, is it possible to obtain a LT T?

<u>Yes</u>, indeed! In NorCal they call it SF, but here we call it L_A:

<u>DEF</u> If A is a $m \times n$ matrix, then $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is defined by:

(Worst airport, though) → $L_A(x) = Ax$ (left multiplication by A)



<u>EX</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $L_A : \mathbb{F}^3 \to \mathbb{F}^2$

$$L_A(x,y,z) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x + 2y + 3z, 4x + 5y + 6z)$$

⚠ Do <u>not</u> confuse A (matrix) with L_A (LT)!

That said, even though they're not the same, we'll see that they share similar properties

PROPERTIES A $M \times N$, $L_A : \mathbb{F}^N \to \mathbb{F}^M$, $L_A(x) = Ax$

$(e_1, \dots, e_N) = \beta$ = STANDARD BASIS OF $\mathbb{F}^N$ $(1,0,0), (0,1,0)$, ETC

γ = STANDARD BASIS OF $\mathbb{F}^M$

(a) $L_A : \mathbb{F}^N \to \mathbb{F}^M$ IS LINEAR

WHY? $L_A(x + cy) = A(x + cy) = Ax + cAy = L_A(x) + c\,L_A(y)$

NOTE IF $y \in \mathbb{F}^M$ AND γ = STANDARD BASIS, THEN $[y]_\gamma = x$

EX $[(2,3)]_\gamma = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ B/C $(2,3) = \underline{\underline{2}}(1,0) + \underline{\underline{3}}(0,1)$

(b) $[L_A]_\beta^\gamma = A$

WHY? j^{TH} COL OF $A = Ae_j = \underbrace{L_A(e_j)}_{\in \mathbb{F}^M} \overset{\text{NOTE}}{=} [L_A(e_j)]_\gamma = j^{TH}$ COL OF $[L_A]_\beta^\gamma$

(c) $L_A = L_B \iff A = B$

WHY? $L_A = L_B \iff [L_A]_\beta^\gamma = [L_B]_\beta^\gamma \overset{(b)}{\iff} A = B$

(d) $L_{A+B} = L_A + L_B$

WHY? $L_{A+B}(x) = (A+B)x = Ax + Bx = L_A(x) + L_B(x)$

(e) IF $T : \mathbb{F}^N \to \mathbb{F}^M$ LT, THEN $T = L_C$ FOR SOME (UNIQUE) MATRIX C

WHY? LET $C = [T]_\beta^\gamma$, THEN

$\underbrace{[L_C(x)]_\gamma}_{} \overset{\text{NOTE}}{=} L_C(x) = Cx \overset{\text{DEF \& NOTE}}{=} [T]_\beta^\gamma [x]_\beta \overset{(a)}{=} [T(x)]_\gamma$

UNIQUENESS IF $T = L_C$ & $T = L_D$, THEN $L_C = L_D \Rightarrow C = D$

(f) $L_{AB} = L_A(L_B)$

$$L_{AB}(e_j) = (AB)(e_j) \overset{(*)}{=} A(Be_j) = A(L_B(e_j)) = L_A(L_B(e_j))$$
$$= (L_A L_B)(e_j)$$

SINCE $\{e_1, \dots, e_N\}$ IS A BASIS OF $\mathbb{F}^N$, GET $L_{AB} = L_A L_B$

PROOF OF (*) (NOT OBVIOUS B/C WE HAVEN'T SHOWN ASSOCIATIVITY OF MATRIX MULTIPLICATION!)

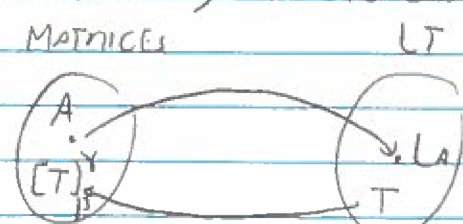
SUPPOSE $B = [\,v_1 \mid \cdots \mid v_p\,]$

THEN
$$(AB)(e_j) = \left(A\,[\,v_1 \mid \cdots \mid v_p\,]\right)(e_j)$$
$$= [\,Av_1 \mid \cdots \mid Av_p\,](e_j)$$
$$= j^{\text{TH}} \text{ COL OF } [\,Av_1 \mid \cdots \mid Av_p\,]$$
$$= Av_j$$
$$= A(j^{\text{TH}} \text{ COL OF } B)$$
$$= A(Be_j) \quad \checkmark$$

(g)
$$I_N = \begin{array}{c} \\ N \end{array}\!\!\overset{N}{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}$$

$L_{I_N} = I_{\mathbb{F}^N}$ $\qquad I_{\mathbb{F}^N}(x) = x$

(WHAT I WANT TO CONVINCE YOU OF IS THAT, IN SOME SENSE, MATRICES ARE LIKE LT, "A IS LIKE L_A")



(GIVEN A, CAN DEFINE L_A.
GIVEN T, CAN DEFINE [T]!)

LASTLY, HERE IS WHY L_A IS SO COOL!

FACT $\quad A(BC) = (AB)C$ (ASSOCIATIVITY)

(BAD WAY: USE DEF OF MATRIX MULTIPLICATION)

OMG WAY

$$\begin{aligned}
L_{A(BC)} &\overset{(1)}{=} L_A(L_{BC}) \\
&\overset{(1)}{=} L_A(L_B(L_C)) \\
&= (L_A L_B) L_C \\
&\overset{(1)}{=} (L_{AB}) L_C \\
&\overset{(1)}{=} L_{(AB)C}
\end{aligned}$$

LT ARE ASSOCIATIVE (B/C FUNCTIONS ARE! $f(gh) = (fg)h$)

$\Rightarrow \quad L_{A(BC)} = L_{(AB)C}$

$\overset{(c)}{\Rightarrow} \quad A(BC) = (AB)C \quad \blacksquare \qquad$ (WOW!)