2.1.12 No, there is no such T. By linearity $T(-2,0,6) = -2 \cdot T(1,0,3)$.

2.1.14a $\Rightarrow$ Assume T does not take linearly independent subsets to linearly independent subsets. This means we can find $v_1, \ldots, v_n$ linearly independent in V, and $a_1, \ldots, a_n$, not all zero, such that $a_1T(v_1) + \cdots + a_nT(v_n) = 0$. By linearity, $a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n) = 0$. Since $v_1, \ldots, v_n$ are linearly independent, $a_1v_1 + \cdots + a_nv_n \neq 0$, so T sends a nonzero vector to the zero vector and T is not one-to-one.

$\Leftarrow$ Assume T sends linearly independent sets to linearly independent sets. Take a basis $v_1, \ldots, v_n$ of V. Given a nonzero $v \in V$ write v as $a_1v_1 + \cdots + a_nv_n$. Then $T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n)$ Since $T(v_1), \ldots, T(v_n)$ are linearly independent and the a_i are not all zero, $T(v)$ is nonzero. Therefore, T is one-to-one.

2.1.16 Since T is linear, to show T is onto, it suffices to show that given the basis $1, x, x^2, x^3, \ldots$, there exists an $f_n \in P(R)$ such that $T(f_n) = x^n$ for all n. If we take $f_n = x^{n+1}/(n+1)$, this works. However, T is not one-to-one because $T(c) = 0$ for all constants c.

2.1.22 Let $a = T(1,0,0), b = T(0,1,0), c = T(0,0,1)$. Then, by linearity, $T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = xa + yb + zc$. For $T : F^n \to F$ you need n scalars, and for $T : F^n \to F^m$ you need n vectors of F^m.

2.1.24 (a) $T(a,b) = (0,b)$

(b) $T(a,b) = T((0, b-a) + (a,a)) = (0, b-a)$.

2.1.28 By linearity $T(0_V) = 0_V$, so $\{0\}$ is T-invariant. By definition, $T(V) \subset V$, so V is T-invariant. $T(N(T)) = \{0\} \subset V$ so N(T) is T-invariant. $T(R(T)) \subset R(T)$ by definition of R(T) so R(T) is T-invariant.

2.1.29 Since W is T-invariant, $T(W) \subset W$, and since W is a subspace, for all $x, y \in W, c \in F, T(x)+T(y) \subset W$ and $cT(x) \in W$ so T_W is well-defined. Since T is a linear transformation, $W \subset V$, and all properties of the linear transformation hold in V, they must hold in W.

2.2.1 (a) True.

(b) True.

(c) False. It is an $n \times m$ matrix.

(d) True.

(e) True.

(f) False.

2.2.2 (a) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$

2.2.4 $[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

2.2.5 (a) $[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(c) $[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

(d) $[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$

(e) $[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$

(f) $[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

(g) $[a]_\gamma = (a)$

2.2.9 $T(az_1 + bz_2) = \overline{az_1 + bz_2} = a\bar{z}_1 + b\bar{z}_2 = a\bar{z}_1 + b\bar{z}_2 = at(z_1) + bT(z_2)$. So T is linear. $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2 · 2.13 Say $aU + bT = 0$. That means for all $v \in V$ $aU(v) + bT(v) = 0_W = U(av) + T(bv)$. Therefore, $U(av) = -T(bv) = T(-bv)$. Since $U(av) \subset R(U)$ and $T(-bv) \subset R(T)$, $U(av) = T(-bv) \subset R(U) \cap R(T) = \{0\}$. Therefore $U(av) = aU(v) = 0 = -bT(v) = T(-bv)$ for all v. Since T and U are nonzero, $a = b = 0$. Thus, U and T are linearly independent.

2 · 2.14 Say $a_1T_1 + \cdots + a_nT_n = 0$. Then $(a_1T_1 + \cdots + a_nT_n)(x) = a_1 = 0$. But also $(a_2T_2 + \cdots + a_nT_n)(x^2) = 2a_1 = 0$, so $a_2 = 0$. Repeating this process until x^n will show that all the a_i are zero so the T_i are linearly independent.

2.3.1 (a) False.

(b) True.

(c) False.

(d) True.

(e) False.

(f) False.

(g) True.

(h) False.

(i) True.

(j) True.

2.3.3 $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$, $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$,

2.3.11 Let V be a vector space and $T : V \to V$ be linear. Prove $T^2 = T_0 \iff R(T) \subseteq N(T)$.

$\Rightarrow$ Take $v \in V$ and consider $T^2(v) = T(T(v))$. Since $T(v) \in R(T)$, by assumption, $T(v) \in N(T)$. Therefore $T^2(v) = T(T(v)) = 0$ for all $v \in V$.

$\Leftarrow$ Take $w in R(T)$. By definition of the range $w = T(v)$ for some $v \in V$. Then $T(w) = T(T(v)) = T^2(v) = T_0(v) = 0$, so $w \in N(T)$. Thus $R(T) \subseteq N(T)$.