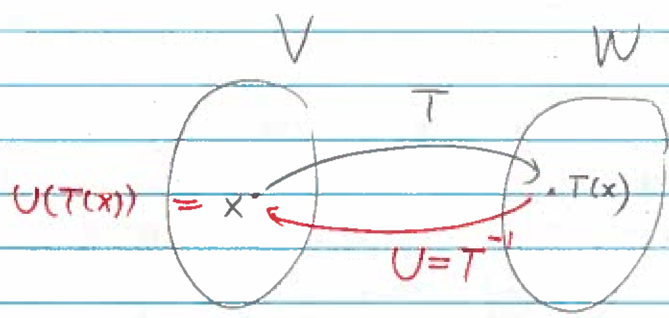
LECTURE 12 – INVERTIBILITY AND ISOMORPHISMS (I) (SECTION 2.4)

:) Now that we've seen how to multiply two matrices, our next order of business is how to find the inverse of a matrix. And yet again, this brings us back to the world of LT.

I – INVERSE OF A LT

<u>PREVIOUSLY</u> ON MATH 13 (NOT SURPRISING, B/C LT ARE <u>FUNCTIONS</u>)



$U = T^{-1}$

<u>DEF</u> $T: V \to W$ is <u>INVERTIBLE</u> if there is $U: W \to V$ such that:
1) $UT = I_V$ $\quad (U(T(x)) = x)$
2) $TU = I_W$ $\quad (T(U(y)) = y)$

(U undoes what T does, <u>EX</u> if T = flight, U = return flight)

<u>IN PARTICULAR</u>
1) $T^{-1}(T(x)) = x$ for all $x \in V$
2) $T(T^{-1}(y)) = y$ for all $y \in W$
3) $T(x) = y \iff T^{-1}(y) = x$ (put T to the right but put a $^{-1}$)

<u>FACT</u> $T^{-1}: W \to V$ is linear

<u>WHY?</u> Let x and $y \in W$

<u>THEN</u> $T(T^{-1}(x) + cT^{-1}(y)) = T(T^{-1}(x)) + cT(T^{-1}(y))$

$$T(T^{-1}(x) + cT^{-1}(y)) = x + cy$$

$$T^{-1}(x) + cT^{-1}(y) = T^{-1}(x + cy) \checkmark$$

CUTE FACTS: 1) $(UT)^{-1} = T^{-1}U^{-1}$ (REVERSE ORDER, SHOE EX)
2) $(T^{-1})^{-1} = T$

Q WHEN IS T INVERTIBLE? (WHEN DOES T^{-1} EXIST?)

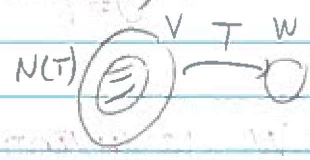
FACT (MATH 13) T INVERTIBLE $\Leftrightarrow$ T IS 1-1 AND ONTO W

IMPORTANT REMARKS

1) IN ORDER FOR $T: V \to W$ TO BE INVERTIBLE, WE MUST HAVE $\dim(V) = \dim(W)$

WHY? • IF $\dim(V) < \dim(W)$, THEN T CANNOT BE ONTO
(SEE HW #4)

• IF $\dim(V) > \dim(W)$, THEN T ≠ 1-1



EX CAN $T: P_2 \to \mathbb{R}^2$ BE INVERTIBLE?
NO $\dim(P_2) = 3 \neq \dim(\mathbb{R}^2) = 2$ (NEVER 1-1)

2) RECALL (2.1) IF $\dim(V) = \dim(W) < \infty$, THEN 1-1 $\Leftrightarrow$ ONTO
(SO USUALLY DON'T HAVE TO SHOW BOTH)

II - INVERSE OF A MATRIX

HOW DOES THAT RELATE TO MATRICES?

DEF A MATRIX A IS INVERTIBLE IF THERE IS SOME MATRIX $B = A^{-1}$ WITH $AB = BA = I$ (B UNDOES A)

IN PARTICULAR $AA^{-1} = A^{-1}A = I$

REMARKS 1) A MUST BE SQUARE $(N \times N)$ (HAS TO DO W/ $DIM(V)=DIM(W)$)

2) SOME INVERSES: $[3]^{-1} = \left[\frac{1}{3}\right]$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

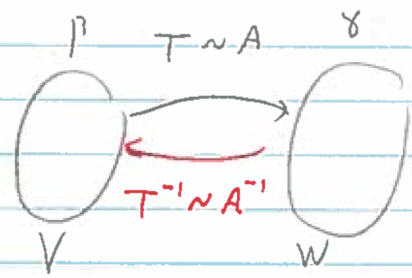
EX IF $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, THEN $A^{-1} = \frac{1}{(1)(3)-(2)(1)}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

CAN CHECK $AA^{-1} = A^{-1}A = I$

SO FAR TODAY WE'VE SEEN TWO DIFFERENT CONCEPTS: THE INVERSE OF A LT AND THE INVERSE OF A MATRIX. IT TURNS OUT THEY'RE TWO DIFFERENT SIDES OF THE SAME COIN!

III - MATRIX OF T^{-1}

PICTURE



(NAMELY: IF A IS THE MATRIX OF T, THEN A^{-1} IS THE MATRIX OF T^{-1})

THEOREM LET $T: V \to W$ LT, β = BASIS OF V, γ = BASIS OF W

LET $A = [T]_\beta^\gamma$ (MATRIX OF T)

THEN 1) T IS INVERTIBLE $\iff$ A IS INVERTIBLE

2) IN BOTH CASES, $[T^{-1}]_\gamma^\beta = A^{-1}$ (REVERSE ORDER)

CONSEQUENCE $[T^{-1}]_\gamma^\beta = \underbrace{\left([T]_\beta^\gamma\right)^{-1}}_{A^{-1}}$

PROOF ($\Rightarrow$) SUPPOSE T IS INVERTIBLE, THEN THERE EXISTS $U = T^{-1} : W \to V$ SUCH THAT $T^{-1}T = I_V$ & $TT^{-1} = I_W$

LET $B = [T^{-1}]_\gamma^\beta$

THE $AB = \underset{A}{[T]_\beta^\gamma} \underset{B}{[T^{-1}]_\gamma^\beta} = [TT^{-1}]_\gamma^\gamma = [I_W]_\gamma^\gamma = I$

SIMILARLY $BA = I$

SO A IS INVERTIBLE AND $A^{-1} \overset{\text{DEF}}{=} B = [T^{-1}]_\gamma^\beta$

($\Leftarrow$) SUPPOSE $A = [T]_\beta^\gamma$ IS INVERTIBLE (A IS $N \times N$)
$M = N$

THEN THERE IS $B = A^{-1}$ WITH $AB = BA = I$

SHOW T IS INVERTIBLE, THAT IS THERE IS $U = T^{-1} : W \to V$ WITH $T^{-1}T = I_V$, $TT^{-1} = I_W$

(IDEA DEFINE T^{-1} SUCH THAT $[T^{-1}]_\gamma^\beta = B = A^{-1}$, THAT IS

$$B = \begin{array}{c|c} & \\ V_1 & \\ V_i & \begin{bmatrix} \mid \\ B_{ij} \\ \mid \end{bmatrix} \\ V_N & \\ & T^{-1}(w_j) \end{array} \Bigg)$$

LET $T^{-1} : W \to V$ BE DEFINED BY :

$$T^{-1}(w_j) = \sum_{i=1}^{N} B_{ij} V_i \quad \text{FOR ALL } j = 1, \ldots, N \quad (M=N)$$

$U = T^{-1}$ EXISTS BY THE LINEAR EXTENSION THEOREM, B/C $\{w_1, \ldots, w_N\}$ IS A BASIS FOR W

THEN $[T^{-1}]_\gamma^\beta = B = A^{-1}$ (BY CONSTRUCTION)

AND $[T^{-1}T]_\beta^\beta = \underbrace{[T^{-1}]_\gamma^\beta}_{B} \underbrace{[T]_\beta^\gamma}_{A} = BA = I = [I_V]_\beta^\beta$

SO $T^{-1}T = I_V$ AND SIMILARLY $TT^{-1} = I_W$, SO T IS INVERTIBLE

MOREOVER, $[T^{-1}]_\gamma^\beta = B = A^{-1} = (A)^{-1} = \left([T]_\beta^\gamma\right)^{-1}$ ∎

EX LET $T: P_2 \to \mathbb{R}^2$ $\quad \beta = \{1, x\}$

$\gamma = \{(1,0),(0,1)\}$

$T(a_0 + a_1 x) = (2a_0 + 3a_1,\ a_0 + 2a_1)$

CAN SHOW $[T]_\beta^\gamma = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

(a) FIND $[T^{-1}]_\gamma^\beta = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

(b) FIND $T^{-1}: \underset{\gamma}{\mathbb{R}^2} \to \underset{\beta}{P_2}$

$[T^{-1}(a_0,a_1)]_\beta = [T^{-1}]_\beta^\gamma [(a_0,a_1)]_\gamma$ (LAST TIME)

$= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad \gamma = \{(1,0),(0,1)\}$

$= \begin{bmatrix} 2a_0 - 3a_1 \\ -a_0 + 2a_1 \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow x \end{matrix}$

WHICH MEANS $T^{-1}(a_0,a_1) = (2a_0 - 3a_1)1 + (-a_0 + 2a_1)x$

$= (2a_0 - 3a_1) + (-a_0 + 2a_1)x$

QUICK REMARK: RECALL $L_A : \mathbb{F}^N \to \mathbb{F}^M \quad L_A(x) = Ax$

FACT $(L_A)^{-1} = (L_{A^{-1}}) \quad (A \ N \times N)$

WHY? $\underbrace{L_A}_{T} \underbrace{L_{A^{-1}}}_{U} = L_{AA^{-1}} = L_I = I_{\mathbb{F}^N}$

$\underbrace{L_{A^{-1}}}_{U} \underbrace{L_A}_{T} = L_{A^{-1}A} = L_I = I_{\mathbb{F}^N}$

SO $L_{A^{-1}} = U = T^{-1} = (L_A)^{-1}$ ∎