MONDAY, APRIL 29, 2019

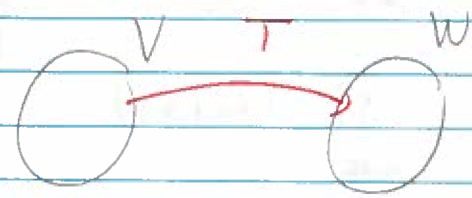
LECTURE 13 – INVERTIBILITY AND ISOMORPHISMS (II) (SECTION 2.4)

TODAY WE WOULD LIKE TO DISCUSS WHAT IT MEANS FOR TWO VS TO "LOOK" THE SAME, AND WE'LL END UP W/ THE MIRACLE OF LA

## I – ISOMORPHIC VS

DEF $T: V \to W$ (LT) IS AN ISOMORPHISM IF $T$ IS 1-1 AND ONTO $W$ ($\Rightarrow T$ INVERTIBLE)

DEF $V$ AND $W$ ARE ISOMORPHIC IF THERE IS AN ISOMORPHISM $T: V \to W$.




EX SHOW $P_2$ AND $\mathbb{F}^3$ ARE ISOMORPHIC

FIND AN ISOMORPHISM $T: P_2 \to \mathbb{F}^3$

DEFINE $T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$

1) SHOW $T$ IS LINEAR (CHECK)

2) SHOW $T$ IS 1-1

IF $T(p) = (0,0,0)$ $(p = a_0 + a_1 x + a_2 x^2)$

THEN $(a_0, a_1, a_2) = (0,0,0) \Rightarrow a_0 = 0, a_1 = 0, a_2 = 0$

SO $p = 0 + 0x + 0x^2 = 0$ ✓

2) SHOW T IS ONTO $\mathbb{F}^3$

( IF $(a_0, a_1, a_2) \in \mathbb{F}^3$, FIND $p$ WITH $T(p) = (a_0, a_1, a_2)$

CHECK $p = a_0 + a_1 x + a_2 x^2$ WORKS )

OR USE $T$ 1-1 AND $\dim(P_2) = \dim(\mathbb{F}^3) = 3$

OTHER WAY FIND $T^{-1} : \mathbb{F}^3 \to P_2$

LET $T^{-1}(a_0, a_1, a_2) = a_0 + a_1 x + a_2 x^2$

CHECK $T^{-1}T = I_{\mathbb{F}^3}$, $TT^{-1} = I_{P_2}$, SO $T^{-1}$ IS AN INVERSE OF $T$

OMG-WAY $\dim(P_2) = \dim(\mathbb{F}^3)$ $(= 3)$

INTERPRETATION $P_2$ IS "LIKE" $\mathbb{F}^3$, $2 + 3x + 4x^2$ "=" $(2, 3, 4)$

## II – ISOMORPHISM AND DIMENSION

( NOW LET ME SHOW YOU WHY THE OMG WAY WORKS )

THEOREM $V$ AND $W$ ARE ISOMORPHIC $\Leftrightarrow$ $\dim(V) = \dim(W)$
(FINITE-DIM)

EX $\mathbb{R}^2$ AND $\mathbb{R}^3$ ARE NOT ISOMORPHIC B/C $\dim(\mathbb{R}^2) \neq \dim(\mathbb{R}^3)$
(MAKES SENSE, A PLANE IS NOT LIKE THE 3D SPACE)

EX $M_{2\times 2}$ AND $\mathbb{R}^4$ ARE ISOMORPHIC B/C $\dim(M_{2\times2}) = \dim(\mathbb{R}^4)$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ "=" $(1, 2, 3, 4)$

PROOF ($\Rightarrow$) SUPPOSE $V$ AND $W$ ARE ISOMORPHIC,

LET $T: V \to W$ BE AN ISOMORPHISM ($= 1\text{-}1$ AND ONTO)

BY RANK-NULLITY, $\dim(N(T)) + \dim(R(T)) = \dim(V)$

SINCE $T$ IS 1-1, $N(T) = \{0\}$, SO $\dim(N(T)) = 0$

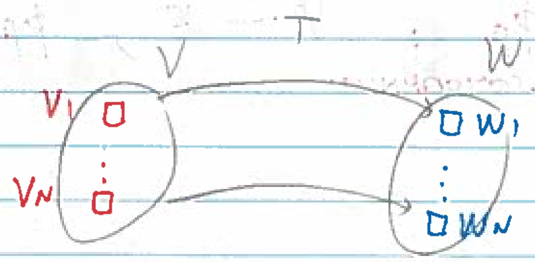
SINCE $T$ IS ONTO, $R(T) = W$

SO WE GET $0 + \dim(W) = \dim(V) \Rightarrow \dim(V) = \dim(W)$ ✓

($\Leftarrow$) SUPPOSE $\dim(V) = \dim(W)$, FIND $T: V \to W$ 1-1 & ONTO

LET $\{v_1, \dots, v_n\}$ BE A BASIS FOR $V$,

$\{w_1, \dots, w_n\}$ BE A BASIS FOR $W$




DEFINE $T$ BY $T(v_j) = w_j$ FOR ALL $j = 1, \dots, n$

$T$ EXISTS BY THE LT EXTENSION THEOREM, SINCE $\{v_1, \dots, v_n\}$ IS A BASIS OF $V$.

MOREOVER, $R(T) = \operatorname{span}\{T(v_1), \dots, T(v_n)\} = \operatorname{span}\{w_1, \dots, w_n\} = W$

SO $T$ IS ONTO $W$

AND SINCE $\dim(V) = \dim(W) < \infty$, $T$ IS ALSO 1-1

SO $T$ IS AN ISOMORPHISM ∎

(ALSO TRUE FOR $\infty$-DIM $V$S, IF YOU REPLACE DIM W/ CARDINALITY)

# III — THE MIRACLES OF LINEAR ALGEBRA

## A — MIRACLE 1 : VS

(AND WITH THIS, WE ARE READY FOR THE TRUE MIRACLES OF LA, THE CULMINATION OF ALL OUR HARD WORK. FOR THE FIRST ONE, REMEMBER THAT VS ARE THESE WEIRD ABSTRACT OBJECTS. IT TURNS OUT THESE OBJECTS ARE NOT ABSTRACT AT ALL; ALL OUR VS ARE $\mathbb{F}^N$ IN DISGUISE)

**MIRACLE #1** ANY FD VS $V$ IS ISOMORPHIC TO $\mathbb{F}^N$ FOR SOME $N$ ($N = \dim(V)$)

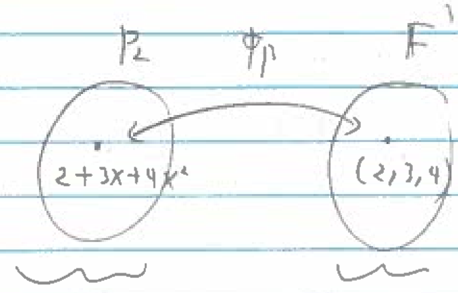
**ISOMORPHISM** LET $\beta = \{v_1, \dots, v_N\}$ BE A BASIS FOR $V$

**FACT** $\phi_\beta : V \to \mathbb{F}^N$, $\phi_\beta(x) = [x]_\beta$ IS AN ISOMORPHISM

(CHECK)

**EX** $V = P_2$ AND $\mathbb{F}^3$ ARE ISOMORPHIC

LET $\beta = \{1, x, x^2\}$, $\phi_\beta(2+3x+4x^2) = [\underline{2}+\underline{3}x+\underline{4}x^2]_\beta = (2,3,4)$




(WITH THE HELP OF COORDINATES, ABSTRACT VS BECOME CONCRETE)

## B – Miracle 2: LT (LT are like matrices)

Miracle #2 $\underbrace{\mathcal{L}(V,W)}_{\text{All LT from } V \text{ to } W}$ is isomorphic to $\underbrace{M_{m\times n}}_{m\times n \text{ matrices}}$ for some m, n

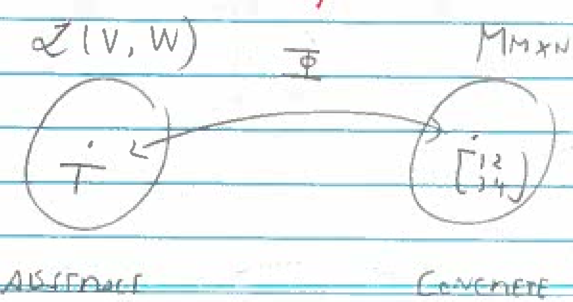
Namely $n = \dim(V)$, $m = \dim(W)$

Consequence $\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W)$ (There are not that many LT!)

Isomorphism Let $\beta = \{v_1, \dots, v_n\}$ be a basis of $V$,
$\gamma = \{w_1, \dots, w_m\}$ be a basis of $W$

Fact $\Phi : \mathcal{L}(V,W) \longrightarrow M_{m\times n}$

$\Phi(T) = [T]_\beta^\gamma$ is an isomorphism




(W/ the help of matrices, abstract LT become concrete)

Why? 1) $\Phi$ linear (check)

2) $\Phi$ 1-1 : Suppose $\Phi(T) = [T]_\beta^\gamma = O$ ← zero matrix
Then $[T]_\beta^\gamma = O = [T_0]_\beta^\gamma$, so
$T = T_0$ ✓

3) $\Phi$ is onto : Let $A \in M_{m\times n}$ be arbitrary
Find $T \in \mathcal{L}(V,W)$ with $\Phi(T) = [T]_\beta^\gamma = A$

**IDEA** LET $T$ BE THE LT WHOSE MATRIX IS $A$

$$\begin{array}{c} w_1 \\ w_i \\ w_m \end{array}\begin{bmatrix} & & \\ & A_{ij} & \\ & & \end{bmatrix}$$
$$T(v_1) \quad T(v_j) \quad T(v_n)$$

**DEFINE** $T \in \mathcal{L}(V, W)$ BY:

$$T(v_j) = A_{1j} w_1 + A_{2j} w_2 + \cdots + A_{mj} w_m = \sum_{i=1}^{m} A_{ij} w_i \in W$$

$T$ EXISTS BY THE LT EXTENSION THEOREM AND SINCE $\{v_1, \ldots, v_n\}$ IS A BASIS OF $V$

AND BY CONSTRUCTION $[T]_\beta^\gamma = A$

SO $\Phi(T) = [T]_\beta^\gamma = A$ ✓

**EX** CALCULATE $(2+3x+4x^2)' = 3+8x$

$$\begin{tikzcd}
P_2 \ni 2+3x+4x^2 \arrow[r, "T(p)=p'"] \arrow[d, "\phi_\beta"'] & 3+8x \in P_1 \\
\mathbb{F}^3 \ni \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \arrow[r, "\Phi(T)=A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}"'] & \begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^2 \arrow[u, "(\phi_\gamma)^{-1}"']
\end{tikzcd}$$

START

END