LECTURE 13 - INVERISIBILITY AND ISOMORPHISMS (II) (SECTION 2.4)

Today we would like to discuss what it means for two VS to "lock" the same, and we'll end up w/ the miracle of LA.

I. Isomorphic VS

DEF. $T : V \rightarrow W$ (LT) is an isomorphism if $T$ is 1-1 and onto $W$ ($\Rightarrow T$ invertible)

DEF. $V$ and $W$ are isomorphic if there is an isomorphism $T : V \rightarrow W$.

EX. Show $P_2$ and $F^3$ are isomorphic

Find an isomorphism $T : P_2 \rightarrow F^3$

DEF. $T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2)$

1) Show $T$ is linear (check).

2) Show $T$ is 1-1.

IF $T(p) = (0, 0, 0)$ ($p = a_0 + a_1x + a_2x^2$)

Then $(a_0, a_1, a_2) = (0, 0, 0) \Rightarrow a_0 = 0, a_1 = 0, a_2 = 0$

So $p = 0 + 0x + 0x^2 = 0$.
(1) \( \text{If } (a_0, a_1, a_2, a_3) \in \mathbb{F}^3, \text{ find } p \text{ with } T(p) = (a_0, a_1, a_2, a_3) \)

\[ p = a_0 + a_1x + a_2x^2 + a_3x^3 \]  \( \text{works.} \)

On we \( T^{-1} \) is 1-1 and \( \text{dim}(P_2) = \text{dim}(\mathbb{F}^3) = 3 \)

Other way \( T^{-1} : \mathbb{F}^3 \rightarrow P_2 \) : add

\[ T^{-1}(a_0, a_1, a_2, a_3) = a_0 + a_1x + a_2x^2 + a_3x^3 \]

Check: \( T^{-1}T = \text{Id} \), \( TT^{-1} = \text{Id} \)

\( \text{Thus } T^{-1} \text{ is an inverse of } T \)

Original way:

\[ \text{dim}(P_2) = \text{dim}(\mathbb{F}^3) = 3 \]

\[ \text{INTERPRETATION}: \ P_2 \text{ like } \mathbb{F}^3, \ 2 + 3x + 4x^2 = (2, 3, 4) \]

II - ISOMORPHISM AND DIMENSION

(Now let me show you why the original way works)

THEOREM: \( V \text{ and } W \text{ are isomorphic } \iff \text{dim}(V) = \text{dim}(W) \)

(Finite-Dim)

EX: \( \mathbb{R}^2 \text{ and } \mathbb{R}^3 \text{ are not isomorphic } \iff \text{dim}(\mathbb{R}^2) \neq \text{dim}(\mathbb{R}^3) \)

(Makes sense. A plane is not like the 3D space)

EX: \( \text{Mat}_2 \text{ and } \mathbb{R}^9 \text{ are isomorphic } \iff \text{dim}(\text{Mat}_2) = \text{dim}(\mathbb{R}^9) \)

\[ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1, 2, 3, 4) \]
PROOF: \((\Rightarrow)\)  
Let \(T: V \to W\) be an isomorphism. \(\quad (=) \quad 1-1\) and onto \(\) 
\[ \begin{align*} 
\text{By Rank-Nullity,} & \quad \dim(N(T)) + \dim(T(V)) = \dim(V) \\
\text{Since} & \quad T \text{ is } 1-1, \quad N(T) = \{0\}, \quad \text{so} \quad \dim(N(T)) = 0 \\
\text{Since} & \quad T \text{ is onto,} \quad \dim(T(V)) = \dim(W) \\
\text{So we get} & \quad 0 + \dim(W) = \dim(V) \Rightarrow \dim(V) = \dim(W) \\
\text{(=) Suppose} & \quad \dim(V) = \dim(W), \quad \text{find} \quad T: V \to W \text{ } 1-1 \text{ and onto} \\
\text{Let} & \quad \{v_1, \ldots, v_n\} \text{ be a basis for } V, \\
\text{and} & \quad \{w_1, \ldots, w_n\} \text{ be a basis for } W. \\
\text{Define} & \quad T \text{ by } T(v_j) = w_j \text{ for all } j = 1, \ldots, n. \\
T \text{ exist by the LT Extension Theorem, since} & \quad \{v_1, \ldots, v_n\} \text{ is a basis of } V. \\
\text{Moreover,} & \quad \dim(T(V)) = \dim(w_1, \ldots, w_n) = W \\
\text{So } & \quad T \text{ is onto } W \\
\text{And since} & \quad \dim(V) = \dim(W) < \infty, \quad T \text{ is also } 1-1 \\
\text{so } & \quad T \text{ is an isomorphism} \quad . \\
\text{(Also true for } \aleph_0 \text{ dimensional, if you replace } \dim W \text{ with cardinality}) \]
III - THE MIRACLES OF LINEAR ALGEBRA

A - MIRACLE 1: VS

(AND WITH THIS, WE ARE READY FOR THE TRUE MIRACLES OF LA, THE CULMINATION OF ALL OUR HARD WORK. FOR THE FIRST ONE, REMEMBER THAT VS ARE THOSE WEIRD ABSTRACT OBJECTS. IT TURNS OUT THESE OBJECTS ARE NOT ABSTRACT AT ALL; ALL OUR VS ARE $\mathbb{F}^n$ IN DISGUISE)

MIRACLE #1 ANY FIELD VS $V$ IS ISOMORPHIC TO $\mathbb{F}^n$.

For some $N$ ( $N = \dim(V)$ )

\[
\text{Let } \mathbf{\beta} = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \text{ be a basis for } V
\]

\[
\text{Fact } \phi_{\mathbf{\beta}} : V \rightarrow \mathbb{F}^n, \quad \phi_{\mathbf{\beta}}(x) = [x]_{\mathbf{\beta}}
\]

( IS AN ISOCONTRPHIC

( CHECK)

EX $V = P_2$ AND $\mathbb{F}^3$ ARE ISOMORPHIC

(LET $\mathbf{\beta} = \{1, x, x^2\}$, $\phi_{\mathbf{\beta}}(2 + 3x + 4x^2) = [2 + 3x + 4x^2]_{\mathbf{\beta}} = (2, 3, 4)$

\[
\begin{array}{ccc}
\mathbb{F}^2 & \phi_{\mathbf{\beta}} & \mathbb{F}^3 \\
| & 2 + 3x + 4x^2 & | (2, 3, 4) \\
\end{array}
\]

ABSTRACT CONCRETE

(WITH THE HELP OF COORDINATE, ABSTRACT VS BECOME CONCRETE)
**B - Miracle 2**: LT (LT are like Matrices.)

Miracle #2: $\mathcal{L}(V, W)$ is isomorphic to $M_{MxN}$ for some $M, N$.

All LT from $V \to W$ namely $N = \text{Dim}(V), M = \text{Dim}(W)$.

**Conclusion**: $\dim(\mathcal{L}(V, W)) = \dim(V) \cdot \dim(W)$.

**Isomorphism**: Let $\mathcal{B} = \{v_1, \ldots, v_m\}$ be a basis of $V$ and $\mathcal{C} = \{w_1, \ldots, w_n\}$ be a basis of $W$.

**Fact**: $\Phi: \mathcal{L}(V, W) \to M_{MxN}$.

$\Phi(T) = [T]_{\mathcal{B}}^{\mathcal{C}}$ is an isomorphism.

$$
\begin{align*}
\mathcal{L}(V, W) & \xrightarrow{\Phi} M_{MxN} \\
\Phi(T) & = [T]_{\mathcal{B}}^{\mathcal{C}}
\end{align*}
$$

With the help of Matrices, the LT become concrete.

**Why?**

1) $\Phi$ Linear (Check)

2) $\Phi$ 1-1: Suppose $\Phi(T) = [T]_{\mathcal{B}}^{\mathcal{C}} = 0$ is a zero matrix.

Then $[T]_{\mathcal{B}}^{\mathcal{C}} = 0 = [T_0]_{\mathcal{B}}^{\mathcal{C}}$, so $T = T_0$.

1) $\Phi$ is onto: Let $A \in M_{MxN}$ be arbitrary.

Find $T \in \mathcal{L}(V, W)$ with $\Phi(T) = [T]_{\mathcal{B}}^{\mathcal{C}} = A$. 
IDEA \quad LET \quad T \quad be \quad the \quad LT \quad with \quad matrix \quad \mathbf{A}:

\[ W_1 \begin{bmatrix} A_{11} \\ \\ W_n \end{bmatrix} \]

\[ T(v_i) \begin{bmatrix} T(v_1) \\ T(v_2) \\ \vdots \\ T(v_n) \end{bmatrix} \]

DEFINE \quad T \in \mathbb{L}(V, W) \ \text{by:}

\[ T(v_i) = A_{ij} w_i + A_{j1} w_1 + \cdots + A_{jn} w_n = \sum_{i=1}^{n} A_{ij} w_i \quad \forall v_i \in V \]

\[ T \text{ exists by the LT Extension Theorem} \quad \text{since} \quad \{v_1, \ldots, v_n\} \]

is a basis of \( V \)

AND BY CONSTRUCTION \quad \mathbf{T} = A

\[ \mathbf{\bar{z}}(T) = \left[ \mathbf{T} \right]_{\mathbf{p}} = A \]

EX \quad Calculate \quad \left( 2 + 3x + 4x^2 \right)' = 3 + 8x

\[ \mathbf{p}_2 \quad T(p) = p' \quad \mathbf{p}_1 \]

\[ \mathbf{T}(T) = \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]

\[ \mathbf{m}_T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]

\[ \mathbf{f}(T) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]