MONDAY, APRIL 22, 2019

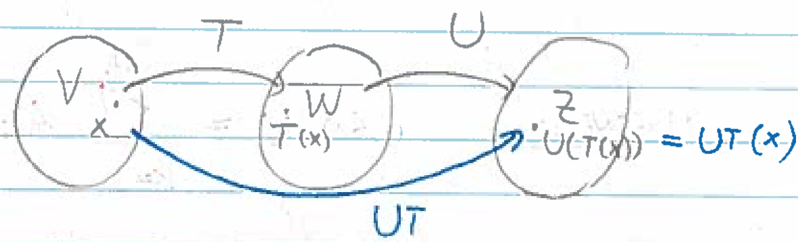
LECTURE 10 – MATRIX MULTIPLICATION (I) (SECTION 2.3)

:) NOW THAT WE KNOW HOW TO ADD MATRICES, OUR NEXT Q IS: HOW DO WE MULTIPLY TWO MATRICES? AND IT'S SOMETHING THAT SEEMED SUPER WEIRD IN MATH 3A, BUT IN THIS CLASS, WE'LL SEE THAT IT MAKES TOTAL SENSE! (B/C THIS COURSE IS SO AWESOME)

I – COMPOSITION

(REMEMBER: SINCE LT ARE FUNCTIONS, ALL OUR FUNCTION LINGO ALSO APPLIES TO LT; IN PARTICULAR WE CAN COMPOSE LT)

SETTING



(IF YOU HAVE LT $T: V \to W$ & $U: W \to Z$, CAN DEFINE $UT: V \to Z$ WHICH BRINGS YOU DIRECTLY FROM V TO Z, = DIRECT FLIGHT)

DEF IF $T: V \to W$ AND $U: W \to Z$ ARE LT, THEN $UT: V \to Z$ IS DEFINED BY:

$$UT(x) = U(T(x))$$

FACT UT IS LINEAR (AND CAN SHOW OTHER PROPERTIES, LIKE $T(U_1 + U_2) = TU_1 + TU_2$)

BUT WHAT DOES THIS HAVE TO DO W/ MATRIX MULT?

II – "REVIEW" OF MATRIX MULTIPLICATION

EX $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 5 & 1 \\ 6 & 1 \end{bmatrix}$, FIND THE $(2,1)^{ST}$ ENTRY OF AB

$$AB = \begin{array}{cc} & 2 \\ 2 & \begin{bmatrix} & \\ \square & \end{bmatrix} \end{array}$$

(TAKE THE SECOND ROW OF A AND THE FIRST COLUMN OF B AND DOT THEM

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = (1)(4) + (2)(5) + (3)(6) = 32$$

$$(AB)_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}$$

$$(AB)_{21} = \sum_{k=1}^{3} A_{2k}B_{k1}$$ (THINK CANCELLING OUT K)

← # COLUMNS OF A

IN GENERAL $$(AB)_{ij} = \sum_{k=1}^{M} A_{ik}B_{kj}$$

(NOW THAT WE HAVE THE FORMULA, LET'S SEE WHERE IT COMES FROM)

III – WHERE THIS COMES FROM

SETTING

$$\begin{tikzcd} V_{\alpha} \arrow[r, "T \sim B"] \arrow[rr, bend right, "UT \sim AB"'] & W_{\beta} \arrow[r, "U \sim A_{\gamma}"] & Z \end{tikzcd}$$

LET $\alpha = \{v_1, \ldots, v_N\}$ BE A BASIS OF V
$\beta = \{w_1, \ldots, w_M\}$ " W
$\gamma = \{z_1, \ldots, z_p\}$ " Z

M OUTPUT INPUT

LET $B = [T]_{\alpha}^{\beta}$ BE THE MATRIX OF T $(M \times N)$
$A = [U]_{\beta}^{\gamma}$ " U $(p \times M)$

WANT TO DEFINE AB SUCH THAT THE MATRIX OF UT IS AB,
THAT IS $[UT]_\alpha^\gamma = AB$

(HOW TO FIND A MATRIX? EVALUATE LT ON YOUR INPUT BASIS AND WRITE IT IN TERMS OF YOUR OUTPUT BASIS)

CONSIDER $(UT)(v_j) \quad (j = 1, \ldots, N)$

GOAL WRITE $(UT)(v_j)$ IN TERMS OF $z_1, \ldots, z_p$ ← OUTPUT BASIS

BUT $(UT)(v_j) = U(T(v_j))$

$$= U\left(\sum_{k=1}^{M} B_{kj} w_k\right)$$

$$\begin{array}{c|c} & T(v_j) \\ \hline w_1 & \\ \vdots & \\ w_k & B_{kj} \\ \vdots & \\ w_M & \end{array} \quad B \; (M \times N)$$

(DEF OF B)

$$= \sum_{k=1}^{M} B_{kj} U(w_k)$$

$$\begin{array}{c|c} & U(w_k) \\ \hline z_1 & \\ \vdots & \\ z_i & A_{ik} \\ \vdots & \\ z_p & \end{array} \quad A \; (p \times M)$$

$$= \sum_{k=1}^{M} B_{kj} \sum_{i=1}^{p} A_{ik} z_i$$

$$= \sum_{k=1}^{M} \sum_{i=1}^{p} B_{kj} A_{ik} z_i$$

$$= \sum_{i=1}^{p} \sum_{k=1}^{M} \left(A_{ik} B_{kj} z_i\right)$$

$$(UT)(v_j) = \sum_{i=1}^{p} \Big(\underbrace{\sum_{k=1}^{M} A_{ik} B_{kj}}_{(AB)_{ij}}\Big) z_i = \sum_{i=1}^{p} (\text{BLAH})\, z_i$$

CONCLUSION $(AB)_{ij} = \sum_{k=1}^{M} A_{ik} B_{kj} \quad (M = \# \text{ COLUMNS OF } A)$

(SO YOU SEE, THIS DEF IS COMPLETELY NATURAL WHEN YOU THINK OF IT IN TERMS OF COMPOSITION)

SUMMARY $[UT]_\alpha^\gamma = \underbrace{[U]_\beta^\gamma}_{A} \underbrace{[T]_\alpha^\beta}_{B}$

NOTE IN GENERAL, $AB \neq BA$, AND THUS B/C IN GENERAL $UT \neq TU$!
(SHOE ANALOGY)

IV - ~~[illegible]~~ TRANSPOSE AND TRACE

(LET'S PRACTICE W/ THIS DEF A BIT)

~~[illegible]~~

DEF A^T IS THE MATRIX WHOSE $(i,j)^{TH}$ ENTRY IS A_{ji}

$$(A^T)_{ij} = A_{ji}$$

EX $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$(1,2)^{TH}$ ENTRY OF A^T IS $(2,1)^{ST}$ ENTRY OF $A = 3$

(SO IT'S LIKE FLIPPING A ON ITS DIAGONAL)

EX SHOW $\underset{M\times N \; N\times P}{(AB)^T} = B^T A^T$ ⚠ REVERSE ORDER!

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{N} A_{jk} B_{ki}$$

$$= \sum_{k=1}^{N} (A^T)_{kj} (B^T)_{ik}$$

$$= \sum_{k=1}^{N} (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij} \checkmark$$

NOTE TRACE

EX $\quad Tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1+5+9 = 15$

$$Tr(A) = \sum_{i=1}^{M} A_{ii}$$

(HW: SHOW $Tr(AB) = Tr(BA)$)

V – SOME QUICK FACTS

1) IF A IS $M \times N$ AND $x \in \mathbb{F}^N$ $\quad {}_M\left[\;A_{ij}\;\right]\left[x_j\right]_N = \left[Ax\right]^M$

THEN $\quad (Ax)_i = \sum_{j=1}^{N} A_{ij}\, x_j$

2) (THEOREM 2.13(a)) IF $B = \left[u_1 \mid u_2 \mid \cdots \mid u_p\right]$, THEN

$$AB = A\left[u_1 \mid u_2 \mid \cdots \mid u_p\right] = \left[Au_1 \mid Au_2 \mid \cdots \mid Au_p\right]$$

EX $\quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix}\right]$

3) (THEOREM 2.13(b)) j^{TH} COLUMN OF A IS Ae_j, $\quad e_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j$

EX $\quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{e_2}$

4) $\quad \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{x} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $\quad$ (Ax IS LINEAR COMBO OF COLUMNS OF A W/ WEIGHTS FROM x)