SUBSET OF S THAT IS

V MOCK MIDTERM

1. *(10 points)* **Find a basis for** $Span(S)$, where

✓ X ✓ X ✓

$$S = \{(1, 0, 0), (2, 0, 0), (1, 1, 1), (2, 1, 1), (1, 1, 0)\}$$

1) ✓ $(1,0,0)$ → KEEP, B/C $\{(1,0,0)\}$ IS LI

2) X $(2,0,0)$ → ELIMINATE B/C $\{(1,0,0),(2,0,0)\}$ IS LD

B/C $(2,0,0) = 2(1,0,0)$

3) ✓ $(1,1,1)$ → KEEP B/C $\{(1,0,0),(1,1,1)\}$ IS LI

(SINCE NEITHER VECTOR IS A MULTIPLE OF THE OTHER

4) X $(2,1,1)$ → ELIMINATE B/C $\{(1,0,0),(1,1,1),(2,1,1)\}$ IS LD

(B/C $(2,1,1) = (1,0,0) + (1,1,1)$)

5) $(1,1,0)$ → KEEP B/C $\{(1,0,0),(1,1,1),(1,1,0)\}$ IS LI

WHY? SUPPOSE $a_1(1,0,0) + a_2(1,1,1) + a_3(1,1,0) = (0,0,0)$

IMPORTANT!

THEN $(a_1+a_2+a_3,\ a_2+a_3,\ a_2) = (0,0,0)$

SO $\begin{cases} a_1+a_2+a_3 = 0 \rightarrow a_1 = -a_2-a_3 = 0 \\ a_2+a_3 = 0 \rightarrow a_3 = -a_2 = 0 \\ a_2 = 0 \end{cases}$

SO $a_1 = a_2 = a_3 = 0$

ANSWER $\{(1,0,0),\ (1,1,1),\ (1,1,0)\}$ (LI AND SPANS SPAN(S),

SO A BASIS FOR SPAN(S))

2. *(30 = 10 + 5 + 15 points)* Let V be a vector space and S be a subset of V

(a) Define what it means for S to be linearly dependent (LD)
(b) Give an example of a set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, but where $\mathbf{v_3}$ is not a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$
(c) Show that S is linearly dependent if and only if there are distinct vectors $\mathbf{u}, \mathbf{u_1}, \cdots, \mathbf{u_n}$ in S such that $\mathbf{u}$ is a linear combination of $\mathbf{u_1}, \cdots, \mathbf{u_n}$

(a) S IS LD IF THERE ARE VECTORS $V_1, \cdots, V_N$ IN S AND CONSTANTS $a_1, \cdots, a_N \in \mathbb{F}$ NOT ALL 0 SUCH THAT

$a_1 V_1 + \cdots + a_N V_N = \underline{0}$, WHERE $\underline{0}$ IS THE ZERO VECTOR IN V

(b) LET $S = \{ \underset{V_1}{(1,0)}, \underset{V_2}{(2,0)}, \underset{V_3}{(0,1)} \}$

THEN S IS LD BECAUSE $\underset{a_1 \; V_1}{2\,(1,0)} + \underset{a_2 \; V_2}{(-1)\,(2,0)} + \underset{a_3 \; V_3}{0\,(0,1)} = \underset{\underline{0}}{(0,0)}$

IMPORTANT: BUT $V_3 = (0,1)$ IS NOT A LINEAR COMBO OF V_1 & V_2 BECAUSE IF

$(0,1) = a_1 (1,0) + a_2 (2,0)$, THEN $(0,1) = (a_1 + 2a_2, 0)$, SO $1 = 0 \Rightarrow \Leftarrow$

$V_3 = a_1 V_1 + a_2 V_2$

(c) ($\Rightarrow$) SUPPOSE S IS LD, THEN THERE ARE VECTORS $V_1, \cdots, V_M$ IN S AND CONSTANTS $a_1, \cdots, a_M$ IN $\mathbb{F}$, NOT ALL ZERO, SUCH THAT

$a_1 V_1 + \cdots + a_M V_M = \underline{0}$

SINCE $a_1, \cdots, a_M$ ARE NOT ALL ZERO, ONE OF THEM, SAY a_i IS NONZERO,

THEN $a_1 V_1 + \cdots + a_{i-1} V_{i-1} + a_i V_i + a_{i+1} V_{i+1} + \cdots + a_M V_M = 0$

$a_i \neq 0$

$a_i V_i = -a_1 V_1 \cdots - a_{i-1} V_{i-1} - a_{i+1} V_{i+1} - a_M V_M$

$\Downarrow$

$V_i = -\frac{a_1}{a_i} V_1 \cdots -\frac{a_{i-1}}{a_i} V_{i-1} - \frac{a_{i+1}}{a_i} V_{i+1} \cdots - \frac{a_M}{a_i} V_M$

LET $N = M-1$, $U = V_i$, $U_1 = V_1, \cdots, U_{i-1} = V_{i-1}$, $U_i = V_{i+1}, \cdots, U_N = V_M$ $(= V_{N+1})$

THEN $$u = -\frac{a_1}{a_i} u_1 \cdots - \frac{a_{i-1}}{a_i} u_{i-1} - \frac{a_{i+1}}{a_i} u_i \cdots - \frac{a_n}{a_i} u_n$$

SO u IS A LINEAR COMBO OF $u_1, \dots, u_n$

($\Leftarrow$) IF THERE ARE VECTORS $u, u_1, \dots, u_n$ IN S SUCH THAT u IS A LINEAR COMBO OF $u_1, \dots, u_n$, THEN THERE ARE CONSTANTS $a_1, \dots, a_n$ SUCH THAT

$$u = a_1 u_1 + \cdots + a_n u_n$$

THEN $$1u - a_1 u_1 \cdots - a_n u_n$$

SINCE NOT ALL OF $\underset{\neq 0}{1}, -a_1, \dots, -a_n$ ARE ZERO,

AND SINCE $u, u_1, \dots, u_n$ ARE VECTORS IN S,

WE GET THAT S IS LD.

3. *(30 = 10 + 20 points)* Let V and W be finite-dimensional vector spaces

(a) Define what it means for V and W to be isomorphic.
(b) Show that V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

(a) V AND W ARE <u>ISOMORPHIC</u> IF THERE EXISTS A LT $T : V \to W$ THAT IS ONE-TO-ONE AND ONTO W.

(b) ($\Rightarrow$) SUPPOSE V & W ARE ISOMORPHIC AND LET $T : V \to W$ BE 1-1 AND ONTO W

THEN BY THE RANK-NULLITY THEOREM,

$$\mathrm{DIM}(N(T)) + \mathrm{RANK}(T) = \mathrm{DIM}(V)$$

SINCE T IS 1-1, $N(T) = \{0_V\}$, SO $\mathrm{DIM}(N(T)) = 0$

SINCE T IS ONTO W, $R(T) = W$, SO $\mathrm{RANK}(T) = \mathrm{DIM}(R(T)) = \mathrm{DIM}(W)$

THEREFORE $0 + \mathrm{DIM}(W) = \mathrm{DIM}(V) \Rightarrow \mathrm{DIM}(V) = \mathrm{DIM}(W)$

($\Leftarrow$) SUPPOSE $\mathrm{DIM}(V) = \mathrm{DIM}(W)$ AND LET'S FIND $T : V \to W$ LT THAT IS 1-1 AND ONTO W.

LET $\beta = \{v_1, \dots, v_n\}$ BE A BASIS OF V AND $\gamma = \{w_1, \dots, w_n\}$ BE A BASIS OF W

DEFINE $T : V \to W$ SUCH THAT <u>$T(v_i) = w_i$</u> FOR ALL $i = 1, \dots, n$

T EXISTS BY THE LINEAR EXTENSION THEOREM

<u>T IS ONTO W</u> BECAUSE $R(T) = \mathrm{SPAN}\{T(v_1), \dots, T(v_n)\}$ (SINCE $\beta = \{v_1, \dots, v_n\}$ IS A BASIS OF V)

$= \mathrm{SPAN}\{w_1, \dots, w_n\}$ (DEF OF T)

$= W$ (SINCE $\gamma = \{w_1, \dots, w_n\}$ IS A BASIS OF W)

<u>T IS 1-1</u> BECAUSE T IS ONTO W AND $\mathrm{DIM}(V) = \mathrm{DIM}(W) < \infty$

SO V & W ARE ISOMORPHIC ∎

4. *(30 points)* Let V and W be finite-dimensional vector spaces, let Z be a subspace of V, and suppose $U : Z \to W$ is linear.

Show that there exists a linear transformation $T : V \to W$ (called an extension of U) such that $T(z) = U(z)$ for all z in Z.

LET $\{v_1, \dots, v_p\}$ BE A BASIS OF Z, WHERE $p = \dim(Z)$

EXTEND $\{v_1, \dots, v_p\}$ TO A BASIS $\{v_1, \dots, v_p, v_{p+1}, \dots, v_N\}$ OF V

$(N = \dim(V))$

DEFINE A LT $T : V \to W$ AS FOLLOWS

$$\begin{cases} T(v_1) = U(v_1) \\ \vdots \\ T(v_p) = U(v_p) \\ T(v_{p+1}) = 0_W \\ \vdots \\ T(v_N) = 0_W \end{cases}$$

THEN T EXISTS BY THE LINEAR EXTENSION THEOREM.

AND BECAUSE $\{v_1, \dots, v_N\}$ IS A BASIS OF V

MOREOVER, IF $z \in Z$, THEN $z = a_1 v_1 + \dots + a_p v_p$ FOR SOME $a_1, \dots, a_p \in \mathbb{F}$

SINCE $\{v_1, \dots, v_p\}$ IS A BASIS OF Z

THEN
$$\begin{aligned} T(z) &= T(a_1 v_1 + \dots + a_p v_p) && \text{T LINEAR} \\ &= a_1 T(v_1) + \dots + a_p T(v_p) && \text{DEF OF } T \\ &= a_1 U(v_1) + \dots + a_p U(v_p) && \text{U LINEAR} \\ &= U(a_1 v_1 + \dots + a_p v_p) \\ &= U(z) \end{aligned}$$

SO $T(z) = U(z)$ FOR ALL $z \in Z$.