1. (10 points) Find a basis for Span(S), where

\[ S = \{ (1, 0, 0), (2, 0, 0), (1, 1, 1), (2, 1, 1), (1, 1, 0) \} \]

1)  \( \checkmark \) \((1, 0, 0)\)  \( \boxed{\text{KEEP}, \text{ B/C } \{ (1,0,0) \} \text{ is LT}} \)

2)  \( \times \) \((2, 0, 0)\)  \( \boxed{\text{ELIMINATE}, \text{ B/C } \{ (1,0,0), (2,0,0) \} \text{ is LD}} \)
\[ \text{B/C } (2,0,0) = 2(1,0,0) \]

3)  \( \checkmark \) \((1, 1, 1)\)  \( \boxed{\text{KEEP}, \text{ B/C } \{ (1, 0, 0), (1, 1, 1) \} \text{ is LT}} \)
\[ \text{(since neither vector is a multiple of the other)} \]

4)  \( \times \) \((2, 1, 1)\)  \( \boxed{\text{ELIMINATE}, \text{ B/C } \{ (1,0,0), (1,1,1), (2,1,1) \} \text{ is LD}} \)
\[ \text{B/C } (2,1,1) = (1,0,0) + (1,1,1) \]

5)  \( \checkmark \) \((1, 1, 0)\)  \( \boxed{\text{KEEP}, \text{ B/C } \{ (1,0,0), (1,1,1), (1,1,0) \} \text{ is LT}} \)
\[ \text{WHY? SUPPOSE } \alpha_1(1,0,0) + \alpha_2(1,1,1) + \alpha_3(1,1,0) = (0,0,0) \]
\[ \text{THEN } (\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3) = (0,0,0) \]
\[ \text{so} \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \rightarrow \alpha_1 = -\alpha_2 - \alpha_3 = 0 \\ \alpha_2 + \alpha_3 = 0 \rightarrow \alpha_3 = -\alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \]
\[ \text{so } \alpha_1 = \alpha_2 = \alpha_3 = 0 \]

\[ \boxed{\text{Answer} \{ (1,0,0), (1,1,1), (1,1,0) \}} \]
\[ \text{(LT and spans Span(S), so a basis for Span(S))} \]
2. \((30 = 10 + 5 + 15 \text{ points})\) Let \(V\) be a vector space and \(S\) be a subset of \(V\)

(a) Define what it means for \(S\) to be linearly dependent

(b) Give an example of a set \(S = \{v_1, v_2, v_3\}\), but where \(v_3\) is not a linear combination of \(v_1\) and \(v_2\)

(c) Show that \(S\) is linearly dependent if and only if there are distinct vectors \(u, u_1, \ldots, u_n\) in \(S\) such that \(u\) is a linear combination of \(u_1, \ldots, u_n\)

\[
\text{(a) } S \text{ is LD if there are vectors } v_1, \ldots, v_n \text{ in } S \text{ and constants } a_1, \ldots, a_n \text{ such that}
\]
\[
a_1 v_1 + \cdots + a_n v_n = 0, \text{ where } 0 \text{ is the zero vector in } V
\]

\[
\text{(b) Let } S = \left\{ (1,0), (2,0), (0,1) \right\}
\]
\[
V_1 \quad V_2 \quad V_3
\]

Then \(S\) is LD because
\[
2(1,0) + (-1)(2,0) + 0(0,1) = (0,0)
\]
\[
a_1 v_1 + a_2 v_2 + a_3 v_3 = 0
\]

\[
\text{but } v_3 = (0,1) \text{ is not a linear combo of } v_1, v_2 \text{ because if}
\]
\[
(0,1) = a_1 (1,0) + a_2 (2,0), \text{ then } (0,1) = (a_1 + 2a_2, 0), \text{ so } 1 = 0 \Rightarrow
\]
\[
v_3 = a_1 v_1 + a_2 v_2
\]

\[
\text{(c) }\text{ Suppose } S \text{ is LD, then there are vectors } v_1, \ldots, v_n \text{ in } S
\]

and constants \(a_1, \ldots, a_n\) in \(V\), not all zero, such that
\[
a_1 v_1 + \cdots + a_n v_n = 0
\]

Since \(a_1, \ldots, a_n\) are not all zero, one of them, say \(a_i\), is nonzero,

Then
\[
a_1 v_1 + \cdots + a_{i-1} v_{i-1} + a_i v_i + a_{i+1} v_{i+1} + \cdots + a_n v_n = 0
\]

\[
 a_i \neq 0
\]

\[
 v_i = \frac{-a_1}{a_i} v_1 \cdots - \frac{a_{i-1}}{a_i} v_{i-1} - \frac{a_{i+1}}{a_i} v_{i+1} - \cdots - \frac{a_n}{a_i} v_n
\]

Let \(N = M^{-1}\), \(U = v_i, U_2 = v_2, \ldots, U_{i-1} = v_{i-1}, U_i = v_{i+1}, \ldots, U_N = v_M (\neq v_{N+1})\)
Then 

\[
U = \frac{-a_1}{a_i} u_1 \ldots \frac{-a_{i-1}}{a_i} u_{i-1} \frac{a_{i+1}}{a_i} u_i \ldots \frac{-a_n}{a_i} u_n
\]

so \(U\) is a linear combo of \(u_1, \ldots, u_n\).

\((\Rightarrow)\) If there are vectors \(u_1, u_2, \ldots, u_n\) in \(S\) such that \(U\) is a linear combo of \(u_1, \ldots, u_n\), then there are constants \(a_1, \ldots, a_n\) such that

\[
U = a_1 u_1 + \ldots + a_n u_n
\]

Then 

\[
1U = a_1 u_1 + \ldots + a_n u_n
\]

since not all of \(\frac{1}{a_i}, \frac{-a_1}{a_i}, \ldots, \frac{-a_n}{a_i}\) are zero.

And since \(u_1, u_2, \ldots, u_n\) are vectors in \(S\),

we get that \(S\) is LD.
3. (30 = 10 + 20 points) Let $V$ and $W$ be finite-dimensional vector spaces.

(a) Define what it means for $V$ and $W$ to be isomorphic.

(b) Show that $V$ and $W$ are isomorphic if and only if $\dim(V) = \dim(W)$.

(a) $V$ and $W$ are isomorphic if there exists a LT $T: V \rightarrow W$ that is one-to-one and onto $W$.

(b) $(\Rightarrow)$ Suppose $V$ and $W$ are isomorphic and let $T: V \rightarrow W$ be 1-1 and onto $W$.

Then by the Rank-nullity theorem,

$$\dim(N(T)) + \text{rank}(T) = \dim(V)$$

Since $T$ is 1-1, $N(T) = \{0\}$, so $\dim(N(T)) = 0$.

Since $T$ is onto $W$, $\text{R}(T) = W$, so $\text{rank}(T) = \dim(N(T)) = \dim(W)$.

Therefore, $0 + \dim(W) = \dim(V) \Rightarrow \dim(V) = \dim(W)$.

$(\Leftarrow)$ Suppose $\dim(V) = \dim(W)$ and let $u$ and $v$ be LTs from $V$ to $W$.

Let $B = \{v_1, \ldots, v_n\}$ be a basis of $V$ and $y = \{w_1, \ldots, w_n\}$ be a basis of $W$.

Define $T: V \rightarrow W$ such that $T(v_i) = w_i$ for all $i = 1, \ldots, n$.

$L$ exists by the linear extension theorem.

$T$ is onto $W$ because $\text{R}(T) = \text{span}\{T(v_1), \ldots, T(v_n)\}$ (since $\beta = \{v_i\}$ is a basis of $V$)

$= \text{span}\{w_1, \ldots, w_n\}$ (def of $T$)

$= W$ (since $\gamma = \{w_1, \ldots, w_n\}$ is a basis of $W$).

$T$ is 1-1 because $T$ is onto $W$ and $\dim(V) = \dim(W) < \infty$.

So $V$ and $W$ are isomorphic.
4. (30 points) Let $V$ and $W$ be finite-dimensional vector spaces, let $Z$ be a subspace of $V$, and suppose $U : Z \rightarrow W$ is linear.

Show that there exists a linear transformation $T : V \rightarrow W$ (called an extension of $U$) such that $T(z) = U(z)$ for all $z$ in $Z$.

Let \( \{ v_1, \ldots, v_p \} \) be a basis of $Z$, where $p = \dim(Z)$.

Extend \( \{ v_1, \ldots, v_p \} \) to a basis \( \{ v_1, \ldots, v_p, v_{p+1}, \ldots, v_n \} \) of $V$ (\( n = \dim(V) \)).

Define $T : V \rightarrow W$ as follows:

\[
\begin{align*}
T(v_1) &= U(v_1) \\
T(v_p) &= U(v_p) \\
T(v_{p+1}) &= 0_w \\
T(v_n) &= 0_w
\end{align*}
\]

Then $T$ exists by the Linear Extension Theorem.

And because \( \{ v_1, \ldots, v_n \} \) is a basis of $V$.

Moreover, if $z \in Z$, then $z = a_1 v_1 + \cdots + a_p v_p$ for some $a_1, \ldots, a_p \in \mathbb{F}$.

Since \( \{ v_1, \ldots, v_p \} \) is a basis of $Z$.

Then $T(z) = T(a_1 v_1 + \cdots + a_p v_p) = a_1 T(v_1) + \cdots + a_p T(v_p)$ by DEF of $T$.

$= a_1 U(v_1) + \cdots + a_p U(v_p)$ \( \cup \) $U$ \( \text{linear} \)$

$= U(a_1 v_1 + \cdots + a_p v_P)$

$= U(z)$

So $T(z) = U(z)$ for all $z \in Z$. 
