WEDNESDAY, MAY 1, 2019

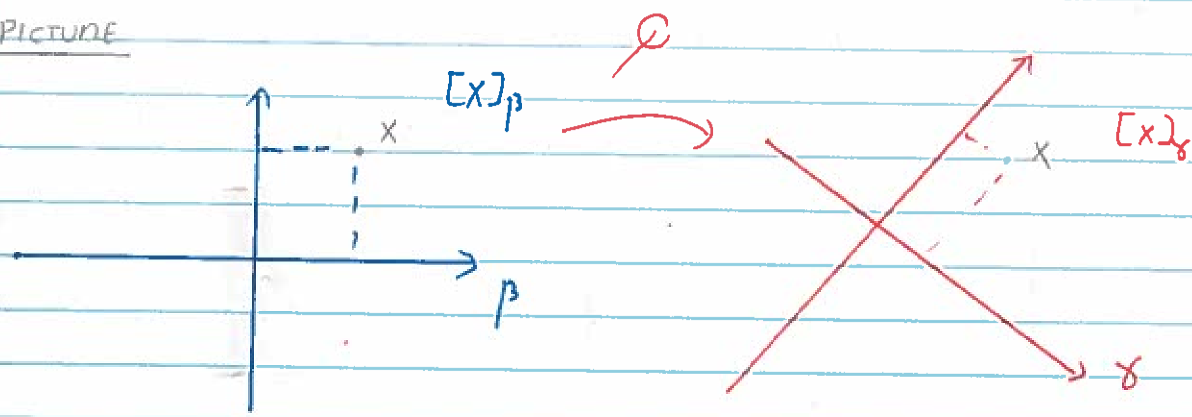
LECTURE 14 — CHANGE OF COORDINATES (SECTION 2.5)

TODAY: INCREDIBLY FUN TOPIC W/ A HUGE NUMBER OF APPLICATIONS: CHANGE OF COORDINATES

I – SETTING

LET β AND γ BE TWO BASES OF V AND $x \in V$

PICTURE



GOAL: FIND A FORMULA FOR $[x]_\gamma$ IN TERMS OF $[x]_\beta$,

DEF: THE CHANGE OF COORDINATES MATRIX $\mathcal{Q} = \underset{\gamma \leftarrow \beta}{\mathcal{Q}}$ FROM β TO γ

IS THE MATRIX SUCH THAT:

$$\boxed{[x]_\gamma = \underset{\gamma \leftarrow \beta}{\mathcal{Q}} [x]_\beta}$$

(LITERALLY FOLLOW THE ARROWS)

II – FORMULA FOR $\mathcal{Q}$

(LET'S FIRST DERIVE A FORMULA FOR $\mathcal{Q}$, THEN I GIVE YOU AN EXAMPLE)

DERIVATION

SINCE $\beta = \{V_1, \dots, V_N\}$ IS A BASIS OF V AND $x \in V$,

$x = x_1 V_1 + \cdots + x_N V_N$ FOR SOME $x_1, \dots, x_N \in \mathbb{F}$

NOTE $[x]_\beta = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$

THEN $\underbrace{[x]_\gamma}_{\text{WTF}} = [x_1 V_1 + \cdots + x_N V_N]_\gamma$

$= x_1 [V_1]_\gamma + \cdots + x_N [V_N]_\gamma$

$= \begin{bmatrix} [V_1]_\gamma & \cdots & [V_N]_\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$

$$[x]_\gamma = \underbrace{\begin{bmatrix} [V_1]_\gamma & \cdots & [V_N]_\gamma \end{bmatrix}}_{{}_\gamma \mathcal{C}_\beta} [x]_\beta$$

DEF $\quad \underset{\gamma \leftarrow \beta}{\mathcal{C}} = \begin{bmatrix} [V_1]_\gamma & \cdots & [V_N]_\gamma \end{bmatrix}$ WHERE $\beta = \{V_1, \dots, V_N\}$

TAKE NOTE YOU TAKE THE OLD VECTORS (IN β) AND EVALUATE THEM WITH THE NEW COORDINATES (γ), NOT THE OTHER WAY AROUND!

EX $\quad V = \mathbb{R}^2$, $\beta = \{\overset{V_1}{(2,4)}, \overset{V_2}{(3,1)}\}$, $\gamma = \{(1,1), (1,-1)\}$

(a) FIND $\underset{\gamma \leftarrow \beta}{\mathcal{C}}$

$[V_1]_\gamma = [(2,4)]_\gamma = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\leadsto$ $(2,4) = a(1,1) + b(1,-1)$

$\Rightarrow \cdots \Rightarrow a = 3, b = -1$

$[v_2]_\gamma = [(3,1)]_\gamma = \begin{bmatrix}2\\1\end{bmatrix}$ $\rightsquigarrow$ $(3,1) = \overbrace{a(1,1) + b(1,-1)}^{\gamma}$

$\rightsquigarrow$ $a = 2,\ b = 1$

$$\mathcal{C} = {}_{\gamma \leftarrow \beta}\mathcal{C} = \begin{bmatrix}3 & 2\\-1 & 1\end{bmatrix}$$

(b) FIND $[x]_\gamma$ GIVEN $[x]_\beta = \begin{bmatrix}3\\2\end{bmatrix}$

$$[x]_\gamma = {}_{\gamma \leftarrow \beta}\mathcal{C}\ [x]_\beta = \begin{bmatrix}3 & 2\\-1 & 1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}13\\-1\end{bmatrix}$$

POINT: AT NO POINT WE ACTUALLY FOUND x! (MORE DIRECT)

NOTE: ${}_{\beta \leftarrow \gamma}\mathcal{C} = \left({}_{\gamma \leftarrow \beta}\mathcal{C}\right)^{-1}$ $\qquad [x]_\beta = \mathcal{C}^{-1}[x]_\beta$

COOL APPLICATION: USE THAT TO CALCULATE

$$\int -3 + \cos(t) + 5\cos^2(t) - 3\cos^3(t) + 7\cos^4(t) + 4\cos^5(t)\, dt$$

(SEE YOUTUBE)

III – CHANGE OF MATRIX

IT TURNS OUT THE SAME THING WE DID TO VECTORS WE CAN DO W/ MATRICES

~~[illegible]~~ LET β & γ BE BASES OF V, AND $T: V \to V$ LT

GOAL: FIND A FORMULA FOR $[T]_\gamma^\gamma$ IN TERMS OF $[T]_\beta^\beta$

DERIVATION: CONSIDER $[T(x)]_\gamma$ (WHERE $x \in V$)

ON THE ONE HAND, $[T(x)]_\gamma = [T]_\gamma^\gamma\,[x]_\gamma$ (DEF OF $[T]_\gamma^\gamma$)

ON THE OTHER HAND, $[T(x)]_\gamma = \underset{\gamma\leftarrow\beta}{Q}\,[T(x)]_\beta = Q\,[T(x)]_\beta$

$= Q\,[T]_\beta^\beta\,[x]_\beta$

$= Q\,[T]_\beta^\beta\,Q^{-1}\,[x]_\gamma$

SO $[T]_\gamma^\gamma\,[x]_\gamma = [T(x)]_\gamma = Q\,[T]_\beta^\beta\,Q^{-1}\,[x]_\gamma$

SINCE x WAS ARBITRARY, GET:

FORMULA $[T]_\gamma^\gamma = Q\,[T]_\beta^\beta\,Q^{-1}$ $\qquad Q = \underset{\gamma\leftarrow\beta}{Q}$

EX $T: \mathbb{F}^2 \to \mathbb{F}^2$, $T(x,y) = (3x-y,\ x+3y)$

LET $\beta = \{(2,4),(3,1)\}$, CAN SHOW $[T]_\beta^\beta = \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}$

LET $\gamma = \{(1,1),(1,-1)\}$, SHOWED $Q = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$

BY FORMULA:

$$[T]_\gamma^\gamma = Q\,[T]_\beta^\beta\,Q^{-1} = \underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}}_{[T]_\beta^\beta}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

(SAYS: IN THE NEW BASIS γ, T IS LIKE $(4x+y,\ -2x+2y)$)

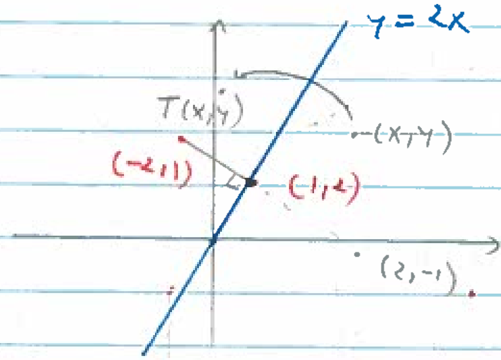
NOTE DEF A AND B ARE SIMILAR IF THERE IS AN INVERTIBLE MATRIX Q WITH $A = QBQ^{-1}$

FORMULA SAYS: $[T]_\gamma^\gamma$ AND $[T]_\beta^\beta$ ARE SIMILAR,

HERE $Q = \underset{\gamma\leftarrow\beta}{Q}$

IV – NEAT APPLICATION

EX LET $T : \mathbb{R}^2 \to \mathbb{R}^2$ BE REFLECTION ABOUT $y = 2x$. FIND A FORMULA FOR T.



PLAN: KNOW WHAT T DOES TO 2 SPECIFIC VECTORS

$T(1,2) = (1,2)$

$T(-2,1) = -(-2,1) = (2,-1)$

LET $\beta = \{(1,2), (-2,1)\}$

$$\left.\begin{array}{l} T(1,2) = (1,2) = \underline{1}(1,2) + \underline{0}(-2,1) \\ T(-2,1) = (2,-1) = \underline{0}(1,2) + \underline{(-1)}(-2,1) \end{array}\right\} \quad [T]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

LET $\gamma = \{(1,0), (0,1)\}$ (STANDARD BASIS)

THEN $Q_{\gamma \leftarrow \beta} = \left[\, [(1,2)]_\gamma \;\; [(-2,1)]_\gamma \,\right] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

THEN $[T]_\gamma^\gamma = Q\, [T]_\beta^\beta\, Q^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$

$$A = [T]_\gamma^\gamma = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

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so $[T(x,y)]_\gamma = [T]_\gamma^\gamma \, [(x,y)]_\gamma$

$$= \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -3x/5 + 4y/5 \\ 4x/5 + 3y/5 \end{bmatrix} \begin{array}{l} \leftarrow (1,0) \\ \leftarrow (0,1) \end{array}$$

$$T(x,y) = \left(-\frac{3x}{5} + \frac{4y}{5}\right)(1,0) + \left(\frac{4x}{5} + \frac{3y}{5}\right)(0,1)$$

$$\boxed{T(x,y) = \left(-\frac{3x}{5} + \frac{4y}{5},\ \frac{4x}{5} + \frac{3y}{5}\right)}$$

WOW, WHO WOULD HAVE THOUGHT?