MONDAY, MAY 6, 2019

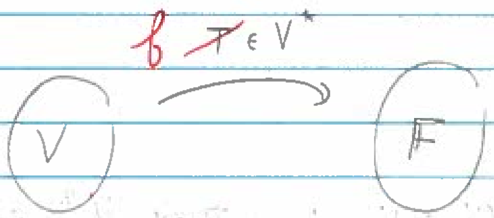
LECTURE 16 – DUAL SPACES (I) (SECTION 2.6)

:) HELLO AND WELCOME TO TODAY'S FREAKY LECTURE ABOUT DUAL SPACES! IF YOU EVER PLAYED LEGEND OF ZELDA, THEN YOU KNOW THAT EVERY WORLD HAS A SHADOW WORLD. THE SAME IS TRUE FOR VS: EVERY VS HAS A SHADOW SPACE, CALLED THE DUAL SPACE

I – DEFINITION

DEF IF V IS A VS, THEN THE DUAL SPACE V^* OF V IS:

$$V^* = \mathcal{L}(V, \mathbb{F}) = \text{SPACE OF ALL LT FROM } V \text{ TO } \mathbb{F}$$



NOTE ELEMENTS IN V^* ARE CALLED LINEAR FUNCTIONALS AND WILL BE DENOTED BY f INSTEAD OF T
(TO EMPHASIZE THAT OUTPUTS ARE SCALARS)

II – EXAMPLES OF LINEAR FUNCTIONALS

EX 1 $V = \mathbb{R}^3$, $f: V \to \mathbb{R}$, $f(x,y,z) = 2x - 3y + 4z$
(LT & VALUES IN $\mathbb{F}$)

EX 2 $V = P_n$, $f: P_n \to \mathbb{R}$, $f(p) = p(1)$

EX $n=2$, $f(1+2x+3x^2) = 1 + 2(1) + 3(1)^2 = 6$

EX 3 $f: M_{n\times n} \to \mathbb{F}$, $f(A) = \mathrm{Tr}(A)$ (LT & VALUES IN $\mathbb{F}$)

EX $n=2$ $f\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1 + 4 = 5$

EX 4 $f : C^{\infty}(\mathbb{R}) \to \mathbb{R}$; $f(\overset{C^{\infty}(\mathbb{R})}{\overset{\downarrow}{g}}) = \int_0^1 g(x)\,dx$

$$f(e^x) = \int_0^1 e^x\,dx = \underbrace{e-1}_{\text{IN } \mathbb{R}}$$

NON-EX 5 $f : \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z) = x^2+y^2+z^2$ NOT LT

NON-EX 6 $f : \mathbb{R}^2 \to \boxed{\mathbb{R}^2}$ $f(x,y) = \underbrace{(x+y,\ 2x-y)}_{\text{NOT IN } \mathbb{F}}$

III – THE MIRACLE:

NOTE: ELEMENTS IN V^* ARE FUNCTIONS (LT FROM V TO $\mathbb{F}$)

(IT SEEMS THAT V & V^* ARE UNRELATED, AND IN FACT IT SEEMS V^* IS MUCH BIGGER THAN V, BUT HERE'S THE MIRACLE:)

FACT (IF $\text{DIM}(V) = N < \infty$), THEN V^* AND V ARE ISOMORPHIC!

(V^* HAS THE SAME SIZE AS V)

WHY? $\text{DIM}(V^*) = \text{DIM}(\mathcal{L}(V, \overset{W}{\mathbb{F}})) = \text{DIM}(V)\,\overset{\overset{1}{\shortparallel}}{\boxed{\text{DIM}(\mathbb{F})}} = \text{DIM}(V)$!

(NOTE WRONG IN ∞ DIM)

(BUT THIS ISN'T SATISFYING! IT WOULD BE NICE TO CONSTRUCT AN EXPLICIT BASIS OF V^*; LUCKILY WE CAN DO THAT)

IV – GRAPHS OF FUNCTIONALS

LET $\beta = \{v_1, \dots, v_n\}$ BE A BASIS OF V AND $f \in V^*$

SINCE f IS A LT, ENOUGH TO KNOW WHAT $f(v_1), \dots, f(v_n)$ ARE

(IN FACT, BECAUSE THE $f(V_j)$ ARE JUST SCALARS, WE CAN REPRESENT f AS A GRAPH)

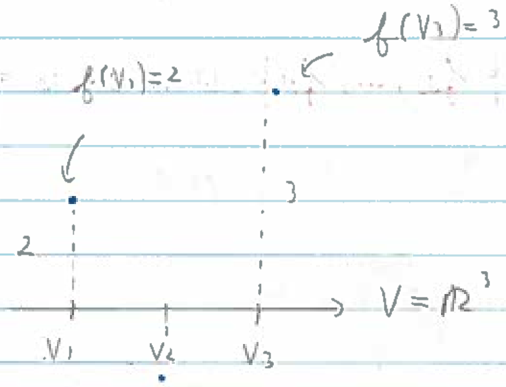
EX $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 2x - y + 3z$

$\beta = \{ \overset{V_1}{(1,0,0)}, \overset{V_2}{(0,1,0)}, \overset{V_3}{(0,0,1)} \}$

$f(V_1) = f(1,0,0) = 2 - 0 + 0 = 2$
$f(V_2) = -1$
$f(V_3) = 3$

GRAPH

$f =$

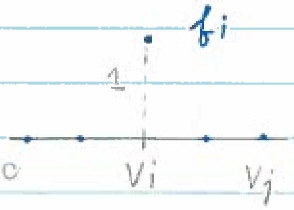


(THIS GRAPHICAL REPRESENTATION MOTIVATES THE DEFINITION OF A DUAL BASIS, WHICH ARE JUST BUILDING BLOCKS OF V^*)

V - DUAL BASIS

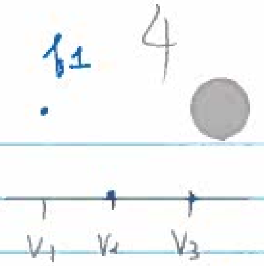
DEF IF $\beta = \{V_1, \dots, V_N\}$ IS A BASIS OF V, THEN THE DUAL BASIS $\beta^* = \{f_1, \dots, f_N\}$ OF V^* IS THE SET OF FUNCTIONALS f_i DEFINED BY:

$$f_i(V_j) = \begin{cases} 1 & \text{IF } j = i \\ 0 & \text{IF } j \neq i \end{cases}$$

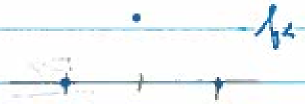


EX (N=3) IF $\beta = \{V_1, V_2, V_3\}$ IS A BASIS OF V
THEN $\beta^* = \{f_1, f_2, f_3\}$, WHERE

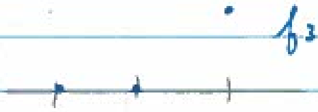
$f_1(V_1) = 1$ $\quad f_1(V_2) = 0$ $\quad f_1(V_3) = 0$



$f_2(V_1) = 0$ $\quad f_2(V_2) = 1$ $\quad f_2(V_3) = 0$



$f_3(V_1) = 0$ $\quad f_3(V_2) = 0$ $\quad f_3(V_3) = 1$



(WHY IMPORTANT ?)

FACT $\beta^* = \{f_1, \dots, f_N\}$ IS A BASIS OF V^*

WHY? SHOW LI : SUPPOSE $a_1 f_1 + \cdots + a_N f_N = f_0 \leftarrow$ ZERO-FUNCTIONAL

THEN FOR ALL $V \in V$

$$(a_1 f_1 + \cdots + a_N f_N)(V) = f_0(V) = \underline{0}$$

$$a_1 f_1(V) + \cdots + a_N f_N(V) = \underline{0} \qquad (*)$$

(RECALL: $\beta = \{V_1, \dots, V_N\}$ IS A BASIS OF V)

LET $\underline{V = V_1}$ IN $(*)$: $\quad a_1 \underbrace{f_1(V_1)}_{1} + a_2 \underbrace{f_2(V_1)}_{0} + \cdots + a_N \underbrace{f_N(V_1)}_{0} = 0$

$$\underline{a_1 = 0}$$

SIMILARLY, LET $V = V_2$ IN $(*)$, GET $\quad a_1(0) + a_2(1) + \cdots + a_N(0) = 0$

$$\underline{a_2 = 0}$$

CONTINUE AND GET $a_1 = 0, \dots, a_N = 0$, SO GET LI ✓

SINCE $\dim(V^*) = \dim(V) = \underline{\underline{N}}$ AND $\{f_1, \dots, f_{\underline{N}}\}$ IS LI, GET SPAN

SO β^* IS A BASIS

WHY COOL? FACT ANY $f \in V^*$ CAN BE WRITTEN (DECOMPOSED) AS $f = a_1 f_1 + \cdots + a_N f_N$

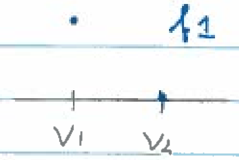
WHERE $a_i = f(V_i)$ ← EASY TO CALCULATE!

(EASY WRITE f IN TERMS OF DUAL VECTORS)

VI - FINDING DUAL BASES

IN PRACTICE, IT'S IMPORTANT TO FIND DUAL BASES EXPLICITLY

EX $V = \mathbb{R}^2$, $\beta = \{\underset{V_1}{(2,1)}, \underset{V_2}{(3,1)}\}$, FIND $\beta^* = \{f_1, f_2\}$



BY DEF $f_1(2,1) = f_1(V_1) = 1$

$f_1(3,1) = f_1(V_2) = 0$

GOAL FIND $f_1(x,y) = f_1(x(1,0) + y(0,1)) \overset{f_1 \text{ LINEAR (!!!)}}{=} x\, f_1(1,0) + y\, f_1(0,1)$

BUT $f_1(2,1) = 1 \Rightarrow 2 f_1(1,0) + f_1(0,1) = 1$ WTF

$f_1(3,1) = 0 \Rightarrow 3 f_1(1,0) + f_1(0,1) = 0$

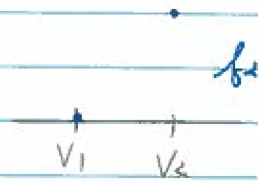
$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} f_1(1,0) \\ f_1(0,1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} f_1(1,0) \\ f_1(0,1) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

SO $f_1(x,y) = x\, f_1(1,0) + y\, f_1(0,1) = x(-1) + y(3) = -x + 3y$

$f_1(x,y) = -x + 3y$

SIMILARLY, USING $f_2(2,1) = f_2(V_1) = 0$

$f_2(3,1) = f_2(V_2) = 1$



GET $f_2(1,0) = 1$, $f_2(0,1) = -2$

SO $f_2(x,y) = x - 2y$

ANS $\beta^* = \{f_1, f_2\}$, $f_1(x,y) = -x + 3y$, $f_2(x,y) = x - 2y$

(BASIS FOR $V^* = (\mathbb{R}^2)^*$)

EX LET $f(x,y) = 2x - 5y$

THEN $f = f(2,1)\, f_1 + f(3,1)\, f_2$

$= (-1)(-x+3y) + (1)(x-2y)$

(EASY TO DECOMPOSE FUNCTION IN TERMS OF DUAL VECTORS)