LECTURE 16 - DUAL SPACES (I) (SECTION 2.6)

HELLO AND WELCOME TO TODAY'S FREAKY LECTURE ABOUT DUAL SPACES!
IF YOU EVER PLAYED LEGEND OF ZELDA, THEN YOU KNOW THAT EVERY WORLD
HAS A SHADOW WORLD. THE SAME IS TRUE FOR VS: EVERY VS HAS A
SHADOW SPACE, CALLED THE DUAL SPACE.

I - DEFINITION

DEF: IF V IS A VS, THEN THE DUAL SPACE V* OF V IS:

V* = \mathcal{L}(V, F) = \text{space of all LT from } V \text{ to } F

NOTE: ELEMENTS IN V* ARE CALLED LINEAR FUNCTIONALS AND WILL
USUALLY BE DENOTED BY f INSTEAD OF T

(EMPHASIZING THAT OUTPUTS ARE SCALARS)

II - EXAMPLES OF LINEAR FUNCTIONALS

EX 1: V = \mathbb{R}^3, f: V \rightarrow \mathbb{R}, f(x, y, z) = 2x - 3y + 4z

(EX 2: V = P_n, f: P_n \rightarrow \mathbb{R}, f(p) = p(1))

EX: n = 2, f(1 + 2x + 3x^2) = 2 + 2(1) + 3(1)^2 = 5

EX 3: f: \text{Mat}_{2\times2} \rightarrow F, f(A) = Tr(A) (LT & VALUE IN F)

EX: n = 2, f \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = 1 + 4 = 5
EX 4  \[ f \cdot c^w(m) = m \cdot j \quad f(g) = \int g(x) \, dx \]

\[ f(e^x) = \int e^x \, dx = e^x - \frac{e^x}{n} \]

Non-Ex 5  \[ f : m^3 \rightarrow m \quad f(x,y,z) = x^2 + y^2 + z^2 \quad \text{Not LT} \]

Non-Ex 5  \[ f : m^3 \rightarrow m^2 \quad f(x,y) = (x+y, 2x-y) \]

III - THE MIRACLE

Note: Elements in \( V^* \) are functions (LT from \( V \) to \( F \))

(\text{It seems that } V \text{ and } V^* \text{ are unrelated, and in fact it seems } \quad \text{\( V^* \) is much bigger than } V, \text{ but here we have the miracle:})

FACT (IF \( \text{dim}(V) = n < \infty \), THEN \( V^* \) AND \( V \) ARE ISOMORPHIC)

(\( V^* \) NOT THE SAME AS \( V \))

 WHY? \[ \text{dim}(V^*) = \text{dim}(\mathcal{L}(V,F)) = \text{dim}(V) \cap \text{dim}(F) = \text{dim}(V) \]

(Note: Wrong in \( \infty \text{ dim} \))

(BUT THIS ISN'T SATISFYING. IT WOULD BE NICE TO CONSTRUCT AN EXPLICIT BASIS OF \( V^* \). LUCKILY WE CAN DO THAT)

IV - GRAPH OF FUNCTIONAL

LET \( f = \{ f(v_1), \ldots, f(v_n) \} \) BE A BASIS OF \( V \) AND \( f \in V^* \)

SINCE \( f \) IS A LT, ENOUGH TO KNOW WHAT \( f(v_1), \ldots, f(v_n) \) ARE
In fact, because the \( \beta(V_i) \) are just scalars, we can represent \( \beta \) as a graph.

**EX**

\[ \beta : \mathbb{R}^3 \to \mathbb{R}, \quad \beta(x, y, z) = 2x - y + 3z \]

\[ V_1 \quad V_2 \quad V_3 \]

\[ \beta = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \} \]

\[ \beta(V_1) = \beta(1, 0, 0) = 2 - 0 + 0 = 2 \]

\[ \beta(V_2) = -1 \]

\[ \beta(V_3) = 3 \]

\[ \beta(V_1) = 2 \quad \beta(V_2) = -1 \quad \beta(V_3) = 3 \]

**Graph**

![Graph](image)

(V = \( \mathbb{R}^3 \))

(The graphical representation motivates the definition of a dual basis, which are just building blocks of \( V^* \)).

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**V - Dual Basis**

**DEF**

If \( \beta = \{ V_1, \ldots, V_n \} \) is a basis of \( V \), then the dual basis \( \beta^* = \{ \beta_1, \ldots, \beta_n \} \) of \( V^* \) is the set of functionals \( \beta_i \) defined by:

\[ \beta_i(V_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \]

**EX**

\( n = 3 \)

If \( \beta = \{ V_1, V_2, V_3 \} \) is a basis of \( V \), then \( \beta^* = \{ \beta_1, \beta_2, \beta_3 \} \), where
\( f_1(v_1) = 1 \quad f_2(v_2) = 0 \quad f_3(v_3) = 0 \)

\( f_1(v_1) = 0 \quad f_2(v_2) = 1 \quad f_3(v_3) = 0 \)

\( f_1(v_1) = 0 \quad f_2(v_2) = 0 \quad f_3(v_3) = 1 \)

(Why important?)

**Fact:** \( \mathbf{B}^* = \{ f_1, \ldots, f_n \} \) is a basis of \( V^* \)

**Why?** Show \( LT \) : Suppose \( a_1 f_1 + \cdots + a_n f_n = f_0 = \text{zero functional} \)

:) Then for all \( v \in V \)

\( (a_1 f_1 + \cdots + a_n f_n)(v) = f_0(v) = 0 \)

\( a_1 f_1(v) + \cdots + a_n f_n(v) = 0 \) \( \tag{*} \)

:) (Recall: \( B = \{ v_1, \ldots, v_n \} \) is a basis of \( V \))

Let \( v = v_1 \) in \((*)\): 
\( a_1 f_1(v_1) + a_2 f_2(v_1) + \cdots + a_n f_n(v_1) = 0 \)

\( a_1 = 0 \)

Similarly, let \( v = v_2 \) in \((*)\), get \( a_2(a_2 + a_1) + a_3 + \cdots + a_n = 0 \)

Continue and get \( a_1 = 0, \ldots, a_n = 0 \), so get \( LT \)

:) Since \( \text{dim}(V^*) = \text{dim}(V) = n \), and \( \{ f_1, \ldots, f_n \} \) is \( LT \), get span...
WHY COOL? FACT ANY $f \in V^*$ CAN BE WRITTEN (DECOMPOSED)

\[ f = a_1 f_1 + \ldots + a_n f_n \]

WHERE $a_i = f(V_i)$ == EASY TO CALCULATE!

VI. FINDING DUAL BASES

IN PRACTICE, IT'S IMPORTANT TO FIND DUAL BASES EXPLICITLY

**EX** $V = \mathbb{R}^2$, $B = \{(2,1), (3,1)\}$, FIND $P^* = \{f_1, f_2\}$

\[ f_1 \quad f_2 \]

**LY DE** $f_1(2,1) = f_1(V_1) = 1$

$- f_1(3,1) = f_1(V_2) = 0$

$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} f_1(1,0) \\ f_1(0,1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} f_1(1,0) \\ f_1(0,1) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

\[ f_1(x,y) = x f_1(1,0) + y f_1(0,1) = x(-1) + y(1) = -x + y \]

\[ f_2(x,y) = -x + 2y \]

**GE**

\[ f_2(2,1) = f_2(V_1) = 0 \]

$- f_2(3,1) = f_2(V_2) = 1$

\[ \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} f_2(1,0) \\ f_2(0,1) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

$\begin{pmatrix} f_2(1,0) \\ f_2(0,1) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

\[ f_2(x,y) = x f_2(1,0) + y f_2(0,1) = x(-1) + y(-1) = -x - y \]

\[ f_2(x,y) = -x - 2y \]

**ANS** $P^* = \{f_1, f_2\}$

\[ f_1(x,y) = -x + 3y, f_2(x,y) = x - 2y \]

(Basis for $V^* = (\mathbb{R}^2)^*$)
EX: Let \( f(x,y) = 2x + 5y \). Then

\[
\begin{align*}
\hat{f} & = \hat{f}(2,1) \hat{f}_1 + \hat{f}(1,1) \hat{f}_1 \\
& = (-1)(-x+2y) + (1)(x-2y)
\end{align*}
\]

(Easy to decompose function in terms of dual vectors.)