SOLUTIONS

MATH 121A – MIDTERM

Name: ______________________

Student ID: ______________________

Instructions: Welcome to your Midterm! You have 50 minutes to take this exam, for a total of 100 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your answer, but also on your work, so please write in complete sentences and explain your steps as much as you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. You'll lose points if you don't sign the statement below. May (the) 3^{rd} be with you!

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: ______________________________

1		10
2		30
3		30
4		30
Total		100

Date: Friday, May 3, 2019.

1. *(10 points)* Let $V = W = P_2(\mathbb{F})$ and define $T : V \to W$ by

$T(p) =$ [redacted] $(p''(x)) + p'(x) + 2p(0)$

Find the matrix $[T]_\beta^\gamma$ of T, where $\gamma = \beta = \{1, x, x^2\}$

Note: In this problem, you do **NOT** need to show your work!

$T(1) = 0 + 0 + 2(1) = 2 = 2(1) + 0x + 0x^2$

$T(x) = 0 + 1 + 2(0) = 1 = 1(1) + 0x + 0x^2$

$T(x^2) = 2 + 2x + 2(0) = 2 + 2x = 2(1) + 2(x) + 0(x^2)$

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

3. *(30 points)* State and prove the Rank-Nullity Theorem (also known as the Dimension Theorem)

RANK – NULLITY THEOREM LET V AND W BE FINITE-DIMENSIONAL VECTOR SPACES AND SUPPOSE $T: V \to W$ IS LINEAR, THEN

$$\dim(N(T)) + \operatorname{rank}(T) = \dim(V)$$

PROOF LET $\{v_1, \dots, v_m\}$ BE A BASIS OF $N(T)$ $(m = \dim(N(T)))$

AND EXTEND $\{v_1, \dots, v_m\}$ TO A BASIS $\{v_1, \dots, v_n\}$ OF V $(n = \dim(V))$

CLAIM $\{T(v_{m+1}), \dots, T(v_n)\}$ IS A BASIS OF $R(T)$, SO $\operatorname{rank}(T) = \dim(R(T)) = n - m$

PROOF 1) SPANS

LET $y \in R(T)$, THEN THERE IS SOME $x \in V$ WITH $y = T(x)$
BUT SINCE $\{v_1, \dots, v_n\}$ IS A BASIS OF V, $v = a_1 v_1 + \dots + a_n v_n$ FOR SOME $a_i \in \mathbb{F}$

THEN $T(x) = T(a_1 v_1 + \dots + a_n v_n) \overset{T \text{ LT}}{=} a_1 \underbrace{T(v_1)}_{0} + \dots + a_m \underbrace{T(v_m)}_{0} + a_{m+1} T(v_{m+1}) + \dots + a_n T(v_n)$

$= a_{m+1} T(v_{m+1}) + \dots + a_n T(v_n)$ $v_1, \dots, v_m \in N(T)$

SO $y = T(x) \in \operatorname{span}\{T(v_{m+1}), \dots, T(v_n)\}$

SINCE y WAS ARBITRARY, WE GET $\operatorname{span}\{T(v_{m+1}), \dots, T(v_n)\} = W$

2) LIN IND

SUPPOSE $a_{m+1} T(v_{m+1}) + \dots + a_n T(v_n) = 0$, THEN $T(a_{m+1} v_{m+1} + \dots + a_n v_n) = 0$ $(T \text{ LT})$

SO $a_{m+1} v_{m+1} + \dots + a_n v_n \in N(T) = \operatorname{span}\{v_1, \dots, v_m\}$

SO $a_{m+1} v_{m+1} + \dots + a_n v_n = a_1 v_1 + \dots + a_m v_m$ FOR SOME $a_1, \dots, a_m \in \mathbb{F}$

SO $-a_1 v_1 - \dots - a_m v_m + a_{m+1} v_{m+1} + \dots + a_n v_n = \underline{0}$, BUT $\{v_1, \dots, v_n\}$ IS A BASIS, SO LI

HENCE $-a_1 = 0 \Rightarrow a_1 = 0, \dots, -a_m = 0 \Rightarrow a_m = 0, a_{m+1} = 0, \dots, a_n = 0$

THEN WE ARE DONE B/C $\operatorname{rank}(T) = n - m = \dim(V) - \dim(N(T)) \Rightarrow \dim(N(T)) + \operatorname{rank}(T) = \dim(V)$

2. *(30 = 10 + 20 points)*

(a) State the Replacement Theorem
(b) Suppose S_1 and S_2 be finite subsets of a vector space V, with S_1 linearly independent, $Span(S_2) = V$, and S_2 has the same number of elements as S_1. Show that S_1 is a basis of V.

(a) LET V BE A VECTOR SPACE,

LIN BE A LINEARLY INDEPENDENT SUBSET OF V WITH M VECTORS

GEN BE A GENERATING SUBSET OF V WITH N VECTORS

THEN: 1) $M \leq N$

2) THERE IS A SUBSET H OF GEN WITH $N-M$ VECTORS SUCH THAT LIN $\cup H$ GENERATES V

(b) IN THE NOTATION ABOVE,

LET LIN $= S_1$ (LI)

GEN $= S_2$ (GENERATES V)

THEN BY THE REPLACEMENT THEOREM THERE IS A SUBSET H OF GEN WITH $|S_2| - |S_1| = 0$ ELEMENTS SUCH THAT

LIN $\cup H = S_1 \cup H$ GENERATES V

BUT SINCE H HAS 0 ELEMENTS, $H = \emptyset$

SO $S_1 \cup H = S_1 \cup \emptyset = S_1$ GENERATES V

SINCE S_1 IS LI AND GENERATES V,

S_1 IS A BASIS OF V

4. *(30 = 5 + 25 points)*

(a) Define: T is an isomorphism from V to W
Note: You do **NOT** need to define the terms that you're using

(b) Let V be the space of 2×2 matrices whose trace is 0. Find an explicit formula of an isomorphism from V to $\mathbb{F}^n$ (for some n of your choice) and show that it's an isomorphism.

(a) $T: V \to W$ IS AN ISOMORPHISM FROM V TO W IF T IS A ONE-TO-ONE LT THAT IS ALSO ONTO T

(ALSO ACCEPTABLE : T IS AN INVERTIBLE LT FROM V TO W)

(b) FIRST OF ALL, NOTICE THAT IF $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ IS IN V,

THEN $Tr(A) = a + d = 0 \Rightarrow d = -a$, SO $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

HENCE $V = \left\{ A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \text{ SUCH THAT } a, b, c \in \mathbb{F} \right\}$

NOW DEFINE $T: V \to \mathbb{F}^3$ (SO $N = 3$) BY

$\boxed{T\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right) = (a, b, c)}$, AND SHOW T LINEAR, 1-1, AND ONTO

T LINEAR

$$T\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix} + k \begin{bmatrix} d & e \\ f & -d \end{bmatrix} \right) = T\left(\begin{bmatrix} a+kd & b+ke \\ c+kf & -a-kd \end{bmatrix} \right)$$

$$= (a+kd, b+ke, c+kf) = (a,b,c) + k(d,e,f)$$

$$= T\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right) + kT\left(\begin{bmatrix} d & e \\ f & -d \end{bmatrix} \right) \checkmark$$

T 1-1

IF $T\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right) = (0,0,0)$, THEN $(a,b,c) = (0,0,0)$
SO $a = 0, b = 0, c = 0$

HENCE $\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to$ ZERO MATRIX

T ONTO

IF $(a,b,c) \in \mathbb{F}^3$, LET $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ ($Tr(A) = 0 \checkmark$)

THEN $T(A) = T\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right) = (a,b,c) \checkmark$, SO T IS ONTO $\mathbb{F}^3$