WEDNESDAY, MAY 8, 2019

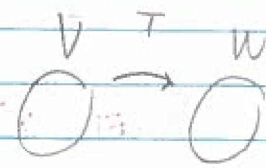
LECTURE 17 – DUAL SPACES (II) (SECTION 2.6)

SO FAR in our matrix adventure, we've seen that matrix multipli-cation comes from composition of LT, and that A^{-1} comes from the inverse of a LT. This begs the Q: where does A^T come from? The answer lies in our dual spaces.

RECALL **DEF** $V^* = \mathcal{L}(V, \mathbb{F})$ = LT from V to $\mathbb{F}$

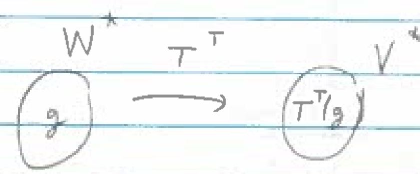
I – THE TRANSPOSE OF A LT

LET $T: V \to W$ be a LT.



(IDEA Just like in A^T we flip the entries of A, for T^T we'd like to "flip" the action of T)

DEFINE: $T^T: W^* \to V^*$



INPUT $g \in W^*$ (so $g: W \to \mathbb{F}$)

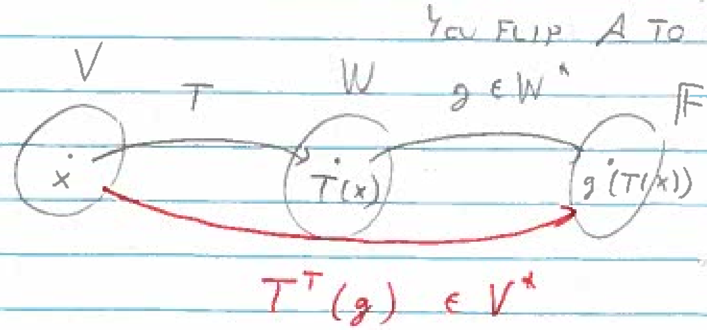
OUTPUT $T^T(g) \in V^*$ (so $T^T(g): V \to \mathbb{F}$)

DEF $T^T: W^* \to V^*$

$$\underbrace{T^T(\overset{W^*}{g})}_{V^*}(x) = g(T(x)) \quad \text{For all } x \in V$$

(You flip T & g, just like you flip A to get A^T)

PICTURE
(g FIXED)



(LET'S SHOW THAT THIS WORKS; NAMELY THAT THE MATRIX OF T^T IS A^T)

II - MATRIX OF T^T

<u>THEOREM</u> LET $T: \overset{\beta}{V} \to \overset{\gamma}{W}$ LT, $A = [T]_\beta^\gamma$

LET $\beta^* = \{f_1, \dots, f_N\}$ BE THE DUAL BASIS OF β (IN V^*)

f_i

(<u>MEANING</u> $f_i(v_j) = \begin{cases} 1 & \text{IF } j = i \\ 0 & \text{IF } j \neq i \end{cases}$ $\quad v_i \quad v_j$)

LET $\gamma^* = \{g_1, \dots, g_M\}$ " γ (IN W^*)

THEN $[T^T]_{\gamma^*}^{\beta^*} = A^T$

$$\begin{array}{ccc} W^* & \xrightarrow{T^T} & V^* \\ \gamma^* & \xrightarrow{A^T} & \beta^* \end{array}$$

<u>WHY?</u>

1) CONSIDER $[T^T]_{\gamma^*}^{\beta^*} = \begin{array}{c|ccc} & T^T(g_1) & T^T(g_j) & T^T(g_M) \\ \hline f_1 & & & \\ \vdots & & & \\ f_N & & & \end{array}$

<u>GOAL</u> CALCULATE $T^T(g_j)$ AND EXPRESS IT IN TERMS OF f_i

(<u>RECALL</u> <u>FACT</u> IF $f \in V^*$, THEN $f = \sum_{i=1}^N f(v_i) f_i$ EASY EXPANSION

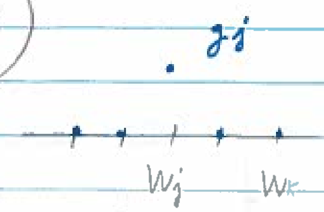
SO $\underbrace{T^T(g_j)}_{\in V^*} = \sum_{i=1}^N T^T(g_j)(v_i) f_i$

$= \sum_{i=1}^N g_j(T(v_i)) f_i$

2) BUT BY DEF OF A, $A = \begin{array}{c|c} & T(v_i) \\ \hline W_1 & \\ W_K & A_{Ki} \\ W_M & \end{array}$

$$T(v_i) = \sum_{k=1}^{M} A_{ki} W_k$$

SO $$g_j(T(v_i)) = g_j\left(\sum_{k=1}^{M} A_{ki} W_k\right)$$



$$= \sum_{k=1}^{M} A_{ki}\, g_j(W_k)$$

$$= A_{1i} \underbrace{g_j(W_1)}_{0} + \cdots + A_{ji} \underbrace{g_j(W_j)}_{1} + \cdots + A_{Mi} \underbrace{g_j(W_M)}_{0}$$

$$= A_{ji}$$

3) HENCE $T^T(g_j) = \sum_{i=1}^{N} A_{ji} f_i = \sum_{i=1}^{N} (A^T)_{ij} f_i$

SO $[T^T]_{\gamma^*}^{\beta^*} = A^T$ ∎

COROLLARY $[T^T]_{\gamma^*}^{\beta^*} = \left([T]_{\beta}^{\gamma}\right)^T$

EX IF $[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, THEN $[T^T]_{\gamma^*}^{\beta^*} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

(SEE BOOK / YOUTUBE FOR A CONCRETE EXPLANATION WHY THIS IS TRUE)

III - V^{**}

(SINCE V^* IS ITSELF A V.S, WE CAN DEFINE ITS DUAL, CALLED V^{**})

DEF $V^{**} = (V^*)^* = \mathcal{L}(V^*, \mathbb{F})$ (LT FROM V^* TO $\mathbb{F}$)

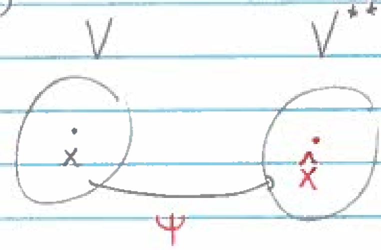
(NOW IF YOU THINK OF V^* IN TERMS OF SHADOWS, YOU MAY ASK: ARE YOU THE SHADOW OF YOUR OWN SHADOW? IN MATH, THE ANSWER IS YES!)

THEOREM IF $\dim(V) < \infty$, THEN V AND V^{**} ARE ISOMORPHIC

WHY? $\dim(V^{**}) = \dim((V^*)^*) = \dim(V^*) = \dim(V)$

BUT WHAT MAKES THIS SO GREAT IS THAT THERE'S A VERY ELEGANT ISOMORPHISM THAT BRINGS YOU FROM V TO V^{**} (JUST LIKE THE MIRROR IN LEGEND OF ZELDA)

ISOMORPHISM

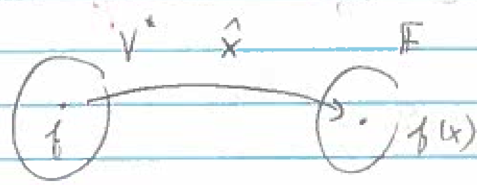


INPUT $x \in V$

OUTPUT $\hat{x} \in V^{**} = \mathcal{L}(V^*, \mathbb{F})$, so $\hat{x} : \overset{f}{V^*} \to \mathbb{F}$

DEF $\hat{x}(\underset{\in V^*}{f}) = f(x)$ "EVALUATION AT x"

PICTURE (FIX x)



(EX $V = \mathbb{R}$, $x = 2$, $\hat{x}(f) = f(2)$)

THEOREM: IF $\dim(V) < \infty$, THEN $\psi : V \to V^{**}$
$\psi(x) = \hat{x}$ IS AN ISOMORPHISM (URSULA)

WHY? 1) ψ LINEAR: IF $x, y \in V$ AND $f \in V^*$, THEN

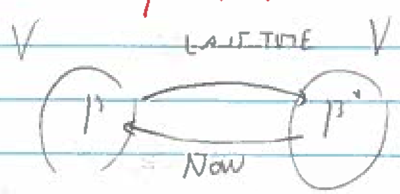
$$\psi(x+cy)(f) = \widehat{x+cy}(f) = f(x+cy) \quad (f \text{ LT})$$
$$= f(x) + c f(y)$$
$$= \hat{x}(f) + c\hat{y}(f)$$
$$= (\hat{x} + c\hat{y})(f)$$
$$= (\psi(x) + c\psi(y))(f)$$

$$\psi(x+cy) = \psi(x) + c\psi(y)$$

2) Ψ 1-1 : SEE NOTES / YT (SEE APPENDIX)

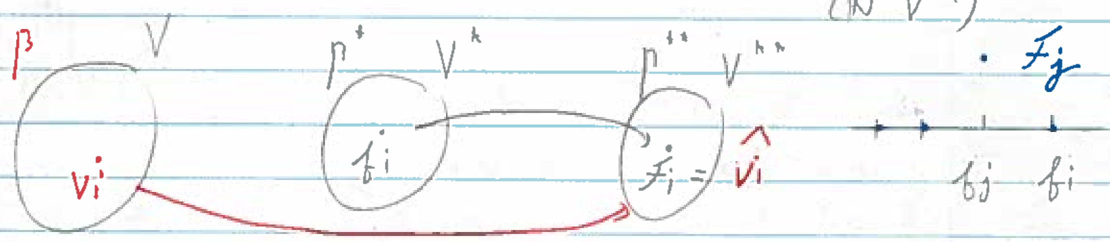
3) Ψ ONTO : B/C $\dim(V^{**}) = \dim(V)$ ∎

COROLLARY EVERY BASIS β^* OF V^* IS THE DUAL BASIS OF SOME BASIS β OF V



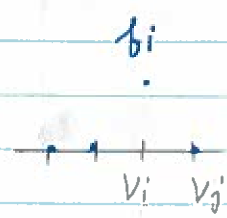
WHY? 1) LET $\beta^* = \{f_1, \dots, f_N\}$ BE A BASIS OF V^* (GOAL FIND β)

LET $\beta^{**} = (\beta^*)^* = \{F_1, \dots, F_N\}$ BE THE DUAL BASIS OF β^* (IN V^{**})



BY THE THEOREM, EACH $F_i = \hat{v}_i$ FOR SOME $v_i \in V$

2) CLAIM $\beta = \{v_1, \dots, v_N\}$ WORKS!



JUST NEED TO CHECK $f_i(v_j) = \begin{cases} 0 & \text{IF } j \neq i \\ 1 & \text{IF } j = i \end{cases}$

BUT $f_i(v_j) = \hat{v}_j(f_i) = F_j(f_i) = \begin{cases} 0 & \text{IF } i \neq j \\ 1 & \text{IF } i = j \end{cases}$ ✓

$j \neq i$

$j = i$

HENCE β^* IS THE DUAL BASIS OF β ∎

EX $V = \mathbb{R}^2$, $\beta^* = \{f_1, f_2\}$, $f_1(x,y) = x - 2y$,
$f_2(x,y) = x + y$

CAN SHOW β^* IS THE DUAL BASIS OF $\beta = \{ \underbrace{\left(\frac{1}{3}, -\frac{1}{3}\right)}_{v_1}, \underbrace{\left(\frac{2}{3}, \frac{1}{3}\right)}_{v_2} \}$

EX $f_1(v_1) = f_1\left(\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{3} - 2\left(-\frac{1}{3}\right) = 1$

$f_1(v_2) = f_1\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3} - 2\left(\frac{1}{3}\right) = 0$

APPENDIX WHY $\Psi(x) = \hat{x}$ IS 1-1

PROOF SUPPOSE $\hat{x} = 0$, SHOW $x = 0$

SINCE $\beta = \{v_1, \dots, v_n\}$ IS A BASIS, $x = a_1 v_1 + \dots + a_n v_n$
FOR SOME $a_i \in \mathbb{F}$

THEN $\hat{x} = 0$

$\Rightarrow (a_1 v_1 + \dots + a_n v_n)^\wedge = 0$

$\Rightarrow a_1 \hat{v}_1 + \dots + a_n \hat{v}_n = 0$

$\Rightarrow (a_1 \hat{v}_1 + \dots + a_n \hat{v}_n)(f) = 0$ FOR ALL $f \in V^*$

$\Rightarrow a_1 \hat{v}_1(f) + \dots + a_n \hat{v}_n(f) = 0$

$\Rightarrow a_1 f(v_1) + \dots + a_n f(v_n) = 0$ $(*)$

NOW LET $f = f_1$ IN $(*)$ TO GET:

$$a_1 \underbrace{f_1(v_1)}_{1} + a_2 \underbrace{f_1(v_2)}_{0} + \dots + a_n \underbrace{f_1(v_n)}_{0} = 0$$

$\Rightarrow$ $a_1 = 0$

SIMILARLY, LET $f = f_2$ TO GET $a_2 = 0$, ETC.

SO $a_1 = 0, \dots, a_n = 0$, HENCE $x = a_1 v_1 + \dots + a_n v_n$
$= 0 v_1 + \dots + 0 v_n = 0$ ∎