FRIDAY, MAY 10, 2019

LECTURE 18 – ELEMENTARY MATRICES ; RANK AND INVERSES (I)
(SECTIONS 3.1 & 3.2)

:) HAPPY F AND WELCOME TO CHAPTER 3! IN THIS CHAPTER, YOU'LL FINALLY UNDERSTAND WHY ALL THE TECHNIQUES IN MATH 3A (ROW-REDUCTION, MATRIX INVERSES) ACTUALLY WORK!

I – ELEMENTARY MATRICES

AND THE KEY TO THIS IS THAT YOU CAN WRITE THE WHOLE PROCESS OF ROW-REDUCTION IN TERMS OF (WHAT ARE CALLED) ELEMENTARY MATRICES.

REVIEW OF ROW OPERATIONS :

① TYPE I INTERCHANGE 2 ROWS OF A :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

ELEMENTARY MATRIX : $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (LOOKS LIKE I, BUT ROWS 1 & 3 ARE FLIPPED)

MEANS :

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

NOTE 1) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (SAME TYPE)

2) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}$ (COLUMN OP)

(2) TYPE 2 MULTIPLY A ROW OF A BY A NUMBER

$$(\times 5)\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 15 & 20 \end{bmatrix}$$

ELEM. MATRIX $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ (LIKE I BUT 5 @ 2^{ND} ROW)

NOTE $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix}$ (SAME TYPE)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 3 & 20 \end{bmatrix}$$

$(\times 5)$ (MULTIPLY 2^{ND} COL BY 5)

(3) TYPE 3 ADD MULTIPLE OF A ROW TO ANOTHER ONE

$$(\times 2)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 12 & 15 \end{bmatrix}$$

ELEM MATRIX

$$\begin{array}{rc} & 1^{ST} \text{ ROW} \\ & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \\ 3^{RD} \text{ ROW} & \end{array}$$

(LIKE I, BUT $(3,1)$ ENTRY IS 2)

NOTE $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (SAME TYPE)

$(\times 2)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 3 \\ 16 & 5 & 6 \\ 25 & 8 & 9 \end{bmatrix}$$

POINT: Row-reduction can be written ~~[illegible]~~ as a product of elementary matrices (this is how we can make row-reduction mathematically rigorous)

EX

$$(\div 2)\begin{bmatrix}2&4\\1&3\end{bmatrix} \to (\times -1)\begin{bmatrix}1&2\\1&3\end{bmatrix} \to \begin{bmatrix}1&2\\0&1\end{bmatrix}$$

start — end

$$\underbrace{\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}1/2&0\\0&1\end{bmatrix}}_{\text{product of elt matrices}}\underbrace{\begin{bmatrix}2&4\\1&3\end{bmatrix}}_{\text{start}} = \underbrace{\begin{bmatrix}1&2\\0&1\end{bmatrix}}_{\text{end}}$$

FACT Elementary matrices are invertible, and their inverses are of the same type (will be useful in the next section)

(That's all I want to say about 3.1; let's move on to 3.2)

II – The rank of a matrix

Now that we know so much about LT, let's see how we can apply our chap 2 knowledge to define the rank of a matrix

Recall
1) If $T: V \to W$, $\text{rank}(T) = \dim(R(T))$
2) If A is a matrix, $L_A: \mathbb{F}^n \to \mathbb{F}^m$ is defined by $L_A(x) = Ax$

DEF If A is $m \times n$, $\text{rank}(A) = \text{rank}(L_A) = \dim(R(L_A))$

(Useful in theory, but we'll see soon how to get a more practical def)
(What's nice about rank, and what makes all of this work, is that applying invertible transformations doesn't affect the rank)

THEOREM If P and Q $(n \times n)$ are invertible, then

1) $\operatorname{rank}(A\underset{\text{inv}}{Q}) = \operatorname{rank}(A)$

WHY? $R(L_{AQ}) = R(L_A L_Q) = L_A \underbrace{L_Q(\mathbb{F}^n)}_{\text{onto}} = L_A(\mathbb{F}^n) = R(L_A)$

$\operatorname{rank}(AQ) = \dim(R(L_{AQ})) = \dim(R(L_A)) = \operatorname{rank}(A)$

2) $\operatorname{rank}(\underset{\text{inv}}{P}A) = \operatorname{rank}(A)$ (skip)

3) $\operatorname{rank}(\underset{\text{inv}}{P}A\underset{\text{inv}}{Q}) = \operatorname{rank}(A)$ (combine 1) & 2))

COROLLARY Row-reduction preserves the rank!

WHY? If B is obtained from A by row-ops, then there is E invertible with $B = EA$

$\operatorname{rank}(B) = \operatorname{rank}(\underset{\text{inv}}{E}A) = \operatorname{rank}(A)$ — Prod of elt matrices

(And it's really this fact that concretely helps us find the rank of a matrix)

III – Column space

DEF $\operatorname{Col}(A)$ = span of columns of A

EX $\operatorname{Col}\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \mathbb{R}^2$

FACT $\operatorname{rank}(A) = \dim(\operatorname{Col}(A))$ (Thm 3.5)

(The one def you're used to from Math 3A)

WHY? $\operatorname{rank}(A) = \operatorname{rank}(L_A) = \dim(R(L_A))$

LET $\beta = \{e_1, \dots, e_n\}$ BE THE STANDARD BASIS OF F^n (THINK $(1,0,0)$ ETC.)

SINCE β IS A BASIS,

$R(L_A) = \text{SPAN}\{L_A(e_1), \dots, L_A(e_n)\}$

$= \text{SPAN}\{A(e_1), \dots, A(e_n)\}$ $\left(\text{EX } \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \underset{\text{COL } 1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right)$

$= \text{SPAN}\{\text{COL } 1 \text{ OF } A, \dots, \text{COL } N \text{ OF } A\}$

$= \text{SPAN}(\text{COLUMNS OF } A)$

$= \text{COL}(A)$

SO $\text{RANK}(A) = \text{DIM}(R(L_A)) = \text{DIM}(\text{COL}(A))$ ∎

EX (LET ME ILLUSTRATE WHY THIS IS SO USEFUL IN FINDING $\text{RANK}(A)$).

$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, FIND $\text{RANK}(A)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{ROW-OPS}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} = B$$

SINCE ROW-OPS ARE RANK-PRESERVING, $\text{RANK}(A) = \text{RANK}(B)$

BUT $\text{RANK}(B) = \text{DIM}(\text{COL}(B))$

$$= \text{DIM}\left(\text{SPAN}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}, \underset{\text{LD}}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}\right\}\right)$$

$$= \text{DIM}\left(\underbrace{\text{SPAN}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}\right\}}_{\text{LI, SO A BASIS}}\right) = 2$$, SO $\text{RANK}(A) = 2$ AS WELL

⚠ IT IS NOT TRUE THAT $Col(A) = Col(B)$!

Ex $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$Col(A) = \text{SPAN}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \neq Col(B) = \text{SPAN}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$