MONDAY, MAY 13, 2019

LECTURE 19 – RANK AND INVERSES (II) (SECTION 3.2)

TODAY ALL ABOUT APPLICATIONS OF RANK, SO YOU CAN SEE WHY THE RANK IS SO USEFUL!

I – THE FUNDAMENTAL THEOREM OF RANK

(FIRST OF ALL, USING ROW AND COLUMN OPERATIONS, YOU CAN TRANSFORM ANY MATRIX INTO A VERY NICE MATRIX, THAT IS ALMOST THE IDENTITY; THINK LIKE ROW-ECHELON FORM, BUT BETTER)

THEOREM IF RANK$(A) = K$, THEN USING ROW AND COLUMN OPS, WE CAN TRANSFORM A INTO D, WHERE

$$D = \begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix} \quad (K \text{ ONES})$$

EX A 4×4, $K = 2$, $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (ALMOST I)

(THE PROOF IS TEDIOUS AND USES INDUCTION, BUT LET ME ILLUSTRATE THIS WITH AN EXAMPLE)

EX

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{(\times -1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

WANT 0   WANT 0 $(\times -2)$ $(\times -1)$

$$\rightarrow \begin{array}{c} \\ (\div -2) \\ (\div -1) \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

$\rightarrow$ (IN THE PROOF, USE INDUCTION ON SMALLER MATRIX)

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{(\times -1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(\times 1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = D$$

NOTE: SINCE ROW/COL OPS ARE RANK-PRESERVING, WE GET
$\text{rank}(A) = \text{rank}(D) = 2.$

II – CONSEQUENCES

(NOW LET'S SEE HOW POWERFUL IT IS)

(FIRST OF ALL, THE NEXT COROLLARY IS JUST RESTATING THE THEOREM IN TERMS OF MATRICES)

COROLLARY 1 IF $\text{rank}(A) = k$ THEN THERE ARE INVERTIBLE MATRICES B & C WITH:

$$\underbrace{B}_{\text{row ops}} A \underbrace{C}_{\text{col ops}} = D = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

COROLLARY 2 $\text{rank}(A^T) = \text{rank}(A)$

(AWESOME PROOF OF THIS ON YOUTUBE USING DUAL SPACES)

WHY? 1) KNOW $BAC = D$

SO $\text{rank}(A) = \text{rank}(\overset{\text{INV}}{B} A \overset{\text{INV}}{C}) = \text{rank}(D)$

2) BUT $BAC = D \Rightarrow (BAC)^T = D^T \Rightarrow C^T A^T B^T = D^T$

SO $\text{rank}(A^T) = \text{rank}(\overset{\text{INV}}{C^T} A^T \overset{\text{INV}}{B^T}) = \text{rank}(D^T)$

3) BUT $\text{rank}(D^T) = \text{rank}(D)$

(EX $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\text{rank}(D) = 2$, $D^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\text{rank}(D^T) = 2$)

4) so $\operatorname{rank}(A^T) = \operatorname{rank}(D^T) = \operatorname{rank}(D) = \operatorname{rank}(A)$ ✓

COROLLARY 3 EVERY INVERTIBLE MATRIX IS A PRODUCT OF ELEMENTARY MATRICES.

WHY? SUPPOSE A IS $N \times N$ AND INVERTIBLE, THEN $\operatorname{rank}(A) = N$

KNOW $BAC = D$, WITH $D = {}_N\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}^{N} = I$, $K = N$ ONES

so $BAC = I$

so $A = B^{-1} I C^{-1} = B^{-1} C^{-1}$

BUT SINCE B & C ARE PROD OF ELEM. MATRICES, SO ARE B^{-1} & C^{-1}, SO IS $B^{-1}C^{-1} = A$ ∎

EX WRITE $\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$ AS A PRODUCT OF ELEM MATRICES

(JUST KEEP TRACK OF ROW-REDUCTION PROCESS)

$$(\times -1)\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \to (\div -3)\begin{bmatrix} 1 & 5 \\ 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} (\times -5) \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} = \left(\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)^{-1} I$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

(USEFUL IN LU-DECOMPOSITION)

III – FINDING A^{-1}

(FINALLY, LET'S USE THE SAME IDEAS TO EXPLAIN WHY THE METHOD IN MATH 3A FOR FINDING INVERSES WORKS:)

EX FIND A^{-1} IF $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{ROW-REDUCE}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right] = [I \mid A^{-1}]$$

SO $A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 3 & -1 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

WHY THIS WORKS?

1) SUPPOSE YOU ROW-REDUCE $[A \mid I]$ TO GET $[I \mid ?]$

THEN $B\,[A \mid I] = [I \mid ?]$ FOR SOME INVERTIBLE B
(B = MATRIX OF ROW-OPS)

2) BUT $B\,[A \mid I] = [BA \mid BI] = [BA \mid B]$

SO $B\,[A \mid I] = [I \mid ?] \Rightarrow [BA \mid \underbrace{BI}_{B}] = [I \mid ?]$

SO $BA = I$ AND $B = ?$

3) BUT $BA = I$ AND B SQUARE $\Rightarrow B = A^{-1}$ (# 10 IN 2.4 ON HW 5)

SO $? = B = A^{-1}$

HENCE $[A \mid I] \rightarrow [I \mid ?] = [I \mid A^{-1}]$

IN FACT THE SAME METHOD ALSO WORKS TO CHECK IF A MATRIX IS INVERTIBLE OR NOT!

EX IS $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 5 & 4 \end{bmatrix}$ INVERTIBLE?

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right]$$

NOT INV!

WHY THIS WORKS?

AGAIN, HAVE $B[A \mid I] = [BA \mid B] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right]$

SO $BA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

SO $\text{rank}(A) = \text{rank}(\underbrace{B}_{\text{INV}}A) = \text{rank}\left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\right) < 3$

SO A IS 3×3 BUT $\text{rank}(A) < 3$, SO A IS NOT INV!

(WHY? IF A IS INV, THEN $\text{rank}(A) = \text{rank}(A\underbrace{A^{-1}}_{\text{INV}}) = \text{rank}(I) = N$)

WHY IS A^{-1} USEFUL?

1) HELPS US SOLVE $Ax = b$ (NEXT TIME)

2) IF $T: V \to W$ IS INVERTIBLE, AND A IS THE MATRIX OF T, THEN A^{-1} IS THE MATRIX OF T^{-1}

$\Rightarrow$ HELPS US FIND A FORMULA FOR T^{-1} (SEE END OF LECTURE 12)