WEDNESDAY, MAY 15, 2019

LECTURE 20 — SYSTEMS OF LINEAR EQUATIONS (I) (SECTIONS 3.3, 3.4)

:) WELCOME TO OUR SYSTEMS OF EQUATIONS - TRILOGY! IN THE NEXT 3 LECTURES, WE'LL FOCUS EXCLUSIVELY ON THE EQUATION $Ax = b$ AND WE'LL SEE WHY ALL THE TECHNIQUES YOU LEARNED IN MATH 3A WORK.

I - REVIEW: LINEAR EQUATIONS

EX SOLVE

$$\begin{cases} x_1 + 3x_2 + 5x_3 = 6 \\ x_1 - 2x_2 + 4x_3 = -8 \\ x_2 - 3x_3 = 0 \end{cases}$$

OF THE FORM $A\underline{x} = \underline{b}$, $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 6 \\ -8 \\ 0 \end{bmatrix}$

$$\underset{\substack{[A \mid b] \\ \text{(AUGMENTED MATRIX)}}}{\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 1 & -2 & 4 & -8 \\ 0 & 1 & -3 & 0 \end{array}\right]} \rightarrow \underset{\substack{\text{(REF)} \\ \text{(PIVOTS = 1, 1, 14)}}}{\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 14 & -14 \end{array}\right]} \rightarrow \underset{\substack{\text{(RREF)} \\ [A' \mid b']}}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

GOAL EXPLAIN WHY THIS WORKS

II - WHY ROW-REDUCTION WORKS

(FIRST OF ALL, WHY DOES ROW-REDUCTION HELP US SOLVE A SYSTEM?)

POINT USING ROW-REDUCTION, WE CAN TURN A COMPLICATED SYSTEM $Ax = b$ INTO AN EASIER SYSTEM $A'x = b'$

EX HERE $A'x = b'$ IS $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 3 \\ x_3 = -1 \end{cases}$

(EASY!)

PROBLEM: HOW DO WE KNOW WE DON'T LOSE ANY SOLUTIONS WHEN ROW-REDUCING?

THEOREM SUPPOSE YOU ROW-REDUCE $[A|b]$ TO GET $[A'|b']$

THEN (x SOLVES $Ax = b$) $\Longleftrightarrow$ (x SOLVES $A'x = b'$)

(SOLUTIONS ARE EXACTLY THE SAME!)

WHY? RECALL THAT ROW-OPS CAN BE WRITTEN IN TERMS OF ELEM. MAT.

SO $B[A|b] = [A'|b']$ FOR SOME INV B

$[BA|Bb] = [A'|b'] \Rightarrow A' = BA, \ b' = Bb$

SO $Ax = b \Rightarrow \underbrace{B(A}_{A'}x) = \underbrace{B(b)}_{b'} \Rightarrow A'x = b'$

CONVERSELY $A'x = b' \Rightarrow BAx = Bb \Rightarrow B^{-1}BAx = B^{-1}Bb$

$\Rightarrow Ax = b$

SO $Ax = b \Longleftrightarrow A'x = b'$ ∎

(SO SINCE SOLUTIONS ARE PRESERVED UNDER ROW-REDUCTION, ALL THAT'S LEFT TO ASK IS: CAN WE TURN ANY MATRIX INTO REF? AND YES WE CAN)

III - ROW-ECHELON FORM

DEF PIVOT = FIRST NONZERO ENTRY IN EACH ROW OF A MATRIX

(NO PIVOT IF ROW IS ALL 0)

EX $\begin{array}{l} \rightarrow \\ \rightarrow \\ \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ PIVOTS = 1, 4

DEF A is in REF if:

1) All the 0 rows are @ the bottom
2) Any entry in the region to the left & below a pivot is 0 (region) (southwest

EX $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Non-Ex $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 4 & 9 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Not 0   Not 0

THEOREM Using row-reduction, can turn any matrix in REF

(Instead of doing the proof, I'll give you an example illustrating the proof)

EX Step 1   Put 0 rows @ bottom by interchanging

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2   Find first $\neq 0$ entry in column 1 and move the row with that entry to the top

row 2 $\rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note If col 1 has all 0's, move on to col 2 and so on

STEP 3 MAKE ALL THE OTHER ENTRIES IN FIRST COLUMN = 0

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} (x - \tfrac{1}{2}) \rightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 1/2 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

WANT = 0

INDUCTION

STEP 4 USE INDUCTION ON SMALLER MATRIX

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

IN REF BECAUSE : 1) ROWS OF 0'S ARE @ THE BOTTOM (BY CONSTRUCTION)

2) ENTRIES TO THE LEFT & BOTTOM OF PIVOTS ARE 0

(FOR COL 1 THIS IS BY CONSTRUCTION, AND FOR OTHER PIVOTS THIS IS BY INDUCTION)

(AND ROW-OPS ON SMALLER MATRIX DON'T AFFECT FIRST ROW OR COLUMN)

IV – REDUCED ROW-ECHELON FORM

PROBLEM REF IS STILL NOT EASY ENOUGH, AND MORE IMPORTANTLY, IT'S NOT UNIQUE!

EX $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$, THEN $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ARE ALL REF OF A

WOULD IDEALLY LIKE SOMETHING THAT'S UNIQUE

DEF A IS IN RREF IF :

1) PIVOTS = 1

2) ALL THE ENTRIES IN EACH PIVOT COLUMN ARE 0 (EXCEPT FOR THE PIVOT)

EX $\begin{bmatrix} 1 & 2 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ RREF

NON-EX $\begin{bmatrix} 1 & 2 & 3 & 3 & 6 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← NOT 0

THEOREM USING ROW-REDUCTION, CAN TURN ANY MATRIX IN RREF

EX STEP 1 TURN A IN REF

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (\div 2)$$

STEP 2 MAKE PIVOTS = 1

← WANT 0

$$\longrightarrow \begin{bmatrix} 1 & 3/2 & 2 & 5/2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (x - 3/2)$$

STEP 3 STARTING WITH THE LAST PIVOT, MAKE OTHER ENTRIES IN THAT PIVOT COLUMN = 0

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ RREF}$$

NOT AFFECTED!

(AND CONTINUE THAT WAY WITH THE NEXT ONE, AND SO FORTH)

FACT RREF OF A MATRIX IS UNIQUE!

(SEE YOUTUBE)