AP SOLUTION

(HW # 7)

AP (a) IF $f(x) = x$ AND $g(x) = x^3$, THEN

$$\hat{f}(g) = \int_{-1}^{1} f(x) g(x)\, dx$$

$$= \int_{-1}^{1} (x)(x^3)\, dx$$

$$= \int_{-1}^{1} x^4\, dx$$

x^4 EVEN

$$= 2\int_{0}^{1} x^4\, dx$$

$$= 2\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{2}{5}$$

(b) $$\hat{f}(g + ch) = \int_{-1}^{1} f(x)\,(g(x) + c\,h(x))\, dx$$

$$= \int_{-1}^{1} f(x) g(x) + c f(x) h(x)\, dx$$

$$= \left(\int_{-1}^{1} f(x) g(x)\, dx\right) + c\left(\int_{-1}^{1} f(x) h(x)\, dx\right)$$

$$= \hat{f}(g) + c\,\hat{f}(h)$$

(c) $\underbrace{T(f+ch)}(g) = \widehat{f+ch}\,(g)$

$$= \int_{-1}^{1} (f(x) + c\,h(x))\, g(x)\, dx$$

$$= \int_{-1}^{1} f(x) g(x) + c \int_{-1}^{1} h(x) g(x)\, dx$$

$$= \hat{f}(g) + c\,\hat{h}(g)$$

$$= (\hat{f} + c\hat{h})(g)$$

$$= \underbrace{(T(f) + c\,T(h))}(g)$$

HENCE $T(f+ch) = T(f) + c\,T(h)$ SINCE g WAS ARBITRARY

ONE-TO-ONE SUPPOSE $T(f) = 0$ ← ZERO FUNCTIONAL

THEN $\hat{f} = 0$, SO $\hat{f}(g) = 0$ FOR ALL g

SO $\int_{-1}^{1} f(x)\, g(x)\, dx = 0$ FOR ALL g

NOW LET $g = f$, THEN GET $\int_{-1}^{1} f(x) f(x)\, dx = 0$

SO $\int_{-1}^{1} \underbrace{(f(x))^2}_{\geq 0} dx = 0$

SO $(f(x))^2 = 0$, SO $f(x) = 0$, SO $f = 0$

SO $T(f) = 0 \Rightarrow f = 0$ SO T IS 1-1

(d) THE HINT TAKE CARE OF MOST OF IT (THANKS, PEYAM ☺)

THE ONLY THING WE NEED TO CHECK IS THAT $\delta(g) = \hat{f}(g)$ LEADS TO A CONTRADICTION

BUT $\delta(g) = g(0) = 0$ (BY DEF OF g)

BUT $\hat{f}(g) = \int_{-1}^{1} f(x) g(x)\, dx$

$g \equiv 0$ OUTSIDE $(x-\varepsilon, x+\varepsilon)$

$$= \int_{x-\varepsilon}^{x+\varepsilon} f(x) g(x)\, dx$$

> 0 BECAUSE $f > 0$ ON $(x-\varepsilon, x+\varepsilon)$ AND $g > 0$ ON $(x-\varepsilon, x+\varepsilon)$

SO IF $\delta(g) = \hat{f}(g)$, THEN WE GET $0 > 0 \Rightarrow \Leftarrow$