2.6.1. (a) F
(b) T
(c) T
(d) T
(e) F
(f) T
(g) T
(h) F

2.6.2. (a) Yes: differentiation is linear, so this function is linear, and it is a map from V to F.
(b) No: a map to V, not F.
(c) Yes: linear, and a map to F.
(d) No: a map to V, not F.
(e) Yes: integration is linear, and it is a map to F.
(f) Yes: linear, and a map to F.

2.6.3. (a) $f_1(1,0,1) = 1$, so $f_1(e_1) + f_1(e_3) = 1$. $f_1(e_1) + 2f_1(e_2) + f_1(e_3) = 0$. $f_1(e_3) = 0$. Thus, $f_1(e_1) = 1$ and $f_1(e_2) = -1/2$, so $f_1(x, y, z) = x - y/2$.

$f_2(e_1) + f_2(e_3) = 0$, $f_2(e_1) + 2f_2(e_2) + f_2(e_3) = 1$, $f_2(e_3) = 0$. Thus, $f_2(e_1) = 0$, and so $f_2(e_2) = -1/2$, so $f_2(x, y, z) = y/2$.

$f_3(e_1) + f_3(e_3) = 0$, $f_3(e_1) + 2f_3(e_2) + f_3(e_3) = 0$, $f_3(e_3) = 1$. Thus, $f_3(e_1) = -1$, $f_3(e_2) = 0$, so $f_3(x, y, z) = -x + z$.

(b) Since β is the standard basis, it is easy to write our functions: $f_1(a_0 + a_1x + a_2x^2) = a_0$, $f_2(a_0 + a_1x + a_2x^2) = a_1$, $f_3(a_0 + a_1x + a_2x^2) = a_2$.

2.6.5. Since we know that V^* is two-dimensional, all we need do is show that f_1 and f_2 are not linearly dependent, i.e. that f_2 is not a scalar multiple of f_1 (it is clear that f_1 is not 0, which is the only other possibility). Let c be any scalar. If $c \neq 2$, let $p(x) = 1$. Then $f_2(p) = 2$ and $f_1(p) = c$, so $f_2 \neq cf_1$. If $c = 2$, let $p(x) = x$. Then $f_2(p) = 2$, but $f_1(p) = 2 * 1/2 = 1$, so $f_2 \neq cf_1$. Thus, for no c is $f_2 = cf_1$, and so they are linearly independent, and thus a basis.

To find a basis for which $\{f_1, f_2\}$ is the dual basis, we let $\{a + bx, c + dx\}$ be the desired basis. Then, since $f_1(a + bx) = 1$, we know that $a + b/2 = 1$. $f_2(a + bx) = 0$, so $2a + 2b = 0$. Thus, $b = -a$, and $a - a/2 = 1$, so $a = 2, b = -2$. By the same method, $c + d/2 = 0$ and $2c + 2d = 1$, so $d/2 = 1$ and thus $d = 2, c = -1$. Thus, our basis is $\{2 - 2x, -1 + 2x\}$.

2.6.11. To show that two functions are equal, we show they are equal on every element in their domains. Let $v \in V$ be any vector. Then $(\psi_2 T)(v)$ and $(T^{tt}\psi_1)(v)$ are both functions from W^* to F. To show that these are equal, again, we show they are equal on every element in their domains. Let g be any element of W^*. By definition of ψ_2, $(\psi_2 T(v))(g) = g(T(v))$. By definition of T^{tt}, $(T^{tt}\psi_1(v))(g) = (\psi_1(v))(T^t(g)) = g(T(v))$. Thus, these two are equal, and so $(\psi_2 T)(v)$ and $(T^{tt}\psi_1)(v)$ are equal. Since this is true for any v, $\psi_2 T$ and $T^{tt}\psi_1$ are equal.

2.6.13. (a) S^0 contains the 0 vector (since the 0 vector is certainly 0 on everything in S). If $f, g \in S^0$, then for any $x \in S$, $(f + g)(x) = f(x) + g(x) = 0$, so $f + g \in S$. Scalar multiplication is the same argument.

(b) Let α be a basis of W. Starting with $\{x\} \cup W$, we complete V to a basis. Then define $f(x) = 1$, $f(y) = 0$ for $y \in \beta$, $y \neq x$. It is easy to check that $f \in V^*$, i.e. that f is linear and maps to F. For $w \in W$, w is a linear combination of elements in α, but we have defined f to be 0 on every element of α, so $f(w) = 0$, and thus $f \in S^0$.

(c) We show both directions of the equality. Let $g \in (S^0)^0$. By Theorem 2.26, g is the image under ψ of some $x \in V$. By the previous part, if $x \notin \text{Span}(S)$, then there is some $f \in S^0$ such that $f(x) \neq 0$. Thus, if $x \notin \text{Span}(S)$, $\psi(x)(f) \neq 0$, and so $\psi(x) \notin (S^0)^0$, since $\psi(x)$ is not 0 on f, which is an element of S^0. Therefore, since $g \in (S^0)^0$, $x \in \text{Span}(S)$. But since ψ is a linear map, if $x \in \text{Span}(S)$, $\psi(x) \in \text{Span}(\psi(S))$. This shows that $(S^0)^0 \subseteq \text{Span}(\psi(S))$.

For the other direction, let $g \in \text{Span}(\psi(S))$. Again by linearity of ψ, we can actually assume that $g = \psi(x)$, where $x \in \text{Span}(S)$. Then, for any element of S^0, h, $g(h) = \psi(x)h = h(x) = 0$, since h is 0 on S, and h is linear. Thus, $\psi(x) \in (S^0)^0$, so $g \in (S^0)^0$, and so $\text{Span}(\psi(S)) \subseteq (S^0)^0$.

2.6.14. Following the hint, we have a basis, $\{x_1, \ldots, x_k\}$ of W (with $k = \dim(W)$), $\beta = \{x_1, \ldots, x_n\}$ an extension to a basis of V (with $n = \dim(V)$), and $\{f_1, \ldots, f_n\}$ the dual basis in W^*, and we wish to show that $\{f_{k+1}, \ldots, f_n\}$ is a basis of W_0. Clearly, $\text{Span}(\{f_{k+1}, \ldots, f_n\}) \subseteq W_0$. Let $g \in W_0$ be any vector. We can write g as $\sum_{i=1}^{k} a_i f_i + \sum_{i=k+1}^{n} b_i f_i$. Then, examining $g(x_1)$, it is easy to see that $a_1 = 0$. We can do this for each $i \leq k$, and get $a_1 = a_2 = \cdots = a_k = 0$. Thus, $g \in \text{Span}(\{f_{k+1}, \ldots, f_n\})$, and so $\{f_{k+1}, \ldots, f_n\}$ span W^0. Since they were constructed to be linearly independent, they are a basis, and so $\dim(W^0) = n - k$. Thus, $\dim(W) + \dim(W^0) = k + n - k = n = \dim(V)$.

2.6.15. We show both directions of the equality. If $g \in N(T^t)$, then $g(Tv) = 0$ for every $v \in V$. Thus, for every $w \in R(T)$, $g(w) = 0$, and so $g \in (R(T))^0$, so $N(T^t) \subseteq (R(T))^0$.

For the other direction, let $g \in (R(T))^0$. Then for every $w \in R(T)$, $g(w) = 0$. Thus, for every $v \in V$, $g(Tv) = 0$, and so $T^t g = 0$, so $g \in N(T^t)$.

2.6.16. We first need to show the simple fact that $L_A^t = L_{A^t}$. This is true since $L_A^t = L_{[L_A^t]} = L_{[L_A]^t} = L_{A^t}$ (these steps are by Theorem 2.25 and Theorem 2.15a). Now this question is easy. Let $n = \dim(V)$. Then $\text{Rank}(L_A) = \dim(R(L_A)) = n - \dim((R(L_A))^0) = n - \dim(N(L_A^t)) = n - \dim(N(L_{A^t})) = \dim(R(L_{A^t}))$.

3.1.1. (a) T
(b) F
(c) T
(d) F
(e) T
(f) F
(g) T
(h) F
(i) T

3.1.2. Adding -2 times column 1 to column 2 transforms A into B. Adding -1 times row 1 to row 2 transforms B into C. Multiplying row 2 by $-1/2$, then subtracting row 1 from row 3, then adding 3 times row 2 from row 1, then subtracting row 3 from row 2, then subtracting 3 times row 3 from row 1 transforms C into I_3.

3.1.3c. Since our matrix is obtained by adding -2 times the first row to the third row, we know that it is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ times I_3. Since $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ is the corresponding inverse elementary operation, and thus the inverse.

3.1.9. To switch row i and row j, add row i to row j, then add -1 times the resulting row j to row i, then add the resulting row i to row j, then multiply row i by -1.

3.2.1. (a) F
(b) F

(c) T
(d) T
(e) F
(f) T
(g) T
(h) T
(i) T

3.2.2eg. For e, add 1/6 column 3 to 1/2 column 5 to get column 4, and multiply column 5 by 2 to get column 2, so the rank is at most 3. Looking at the last 3 rows, it is clear that they are linearly independent, so the rank is 3. For g, by inspection on the columns, it is clear that the rank is 1.

3.2.3. The rank is the dimension of the span of the columns. If the dimension is 0, then every column vector is in the span of the empty basis, and thus is the zero vector. Since every column vector is 0, the matrix is the 0 matrix.

3.2.4a. Subtracting row 1 from row 3, then subtracting column 2 from column 1, then switching column 1 and column 2, then subtracting the appropriate multiples from the third and fourth columns to zero them, gives the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, showing that the rank is 2.

3.2.5c. The rank is 2:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -2 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \to$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -2 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 9 & -6 & 4 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -2 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right)$$

3.2.6a. $T(1) = -1$, $T(x) = 2-x$, and $T(x^2) = 2+4x-x^2$. These are linearly independent, so T is invertible. Its inverse takes 1 to -1, x to $-x-2$, and x^2 to $-x^2-4x-10$.

3.2.7. We note that to reach I_3, we first subtract the second row from the first, divide the first by 2, subtract the second row from the third, subtract the first row from the third, then the third from the second, then switch the first and second. Thus, this matrix is the product of the inverses of the above operations:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$