FRIDAY, MAY 17, 2019

LECTURE 21 – SYSTEMS OF LINEAR EQUATIONS (II) (SECTIONS 3.3 & 3.4)

HAPPY F AND LET'S CONTINUE OUR SYSTEMS OF EQUATIONS – EXTRAVAGANZA!

I – NUL(A), COL(A), RANK(A)

(AND LET'S START WITH A VERY CLASSICAL MATH 3A EXERCISE AND EXPLAIN WHY IT WORKS)

EX

$$A = \begin{bmatrix} 2 & -6 & 9 & 0 & 0 \\ 3 & -4 & 7 & 2 & 0 \\ 2 & -6 & 6 & -6 & 1 \end{bmatrix} \sim A' = \begin{array}{c} \begin{array}{ccccc} W1 & & W3 & & W5 \end{array} \\ \begin{bmatrix} 1 & -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \text{ (RREF)}$$

(a) FIND RANK(A)

CLAIM: RANK(A) = # PIVOTS OF A

WHY? SINCE ROW-REDUCTION PRESERVES RANK, RANK(A) = RANK(A')

BUT RANK(A') = DIM (COL(A'))

$$= \text{DIM}\left(\text{SPAN}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \underbrace{\cancel{\begin{bmatrix}-2\\0\\0\end{bmatrix}}}_{LD}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \underbrace{\cancel{\begin{bmatrix}-4\\2\\0\end{bmatrix}}}_{LD}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$$

$$\left(\text{BUT } \begin{bmatrix}-2\\0\\0\end{bmatrix} = (-2)\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-4\\2\\0\end{bmatrix} = (-4)\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

(ALWAYS TRUE FOR RREF; NON-PIVOT COLS ARE LINEAR COMBOS OF PIVOT COLS)

$$= \text{DIM}\left(\underbrace{\text{SPAN}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}}_{LI}\right)$$

$$= 3$$

SO RANK(A) = 3 = # PIVOTS

(b) FIND A BASIS FOR $Col(A)$

CLAIM: PIVOT COLUMNS OF $\underline{A}$ FORM A BASIS FOR $Col(A)$

HERE $\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ IS A BASIS FOR $Col(A)$ (COLS 1, 3, 5 OF A)

WHY? 1) BECAUSE $\dim(Col(A)) = rank(A) = 3$, ENOUGH TO FIND 3 LI VECTORS IN $Col(A)$ (*)

2) KNOW $W_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $W_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $W_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ LI

(ALWAYS TRUE FOR RREF, PIVOT COLS OF A' ARE LI)

3) ALSO KNOW $A' = BA$ FOR SOME INV B

$$[W_1 \; W_2 \cdots W_5] = B[V_1 \cdots V_5]$$

(W_j = COL j OF A', V_j = " A)

$$[W_1 \cdots W_5] = [BV_1 \cdots BV_5]$$

SO $W_j = BV_j$ FOR ALL j

IN PARTICULAR, $\{BV_1, BV_3, BV_5\} = \{W_1, W_3, W_5\}$ LI

FACT: IF B IS INV, $\{BV_1, BV_3, BV_5\}$ LI $\Leftrightarrow$ $\{V_1, V_3, V_5\}$ LI

SO $\{V_1, V_3, V_5\} = \left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ IS LI, SO A BASIS FOR $Col(A)$

(PIVOT COLS OF A) (BY (*))

(c) FIND $\dim(\text{NUL}(A))$

BY RANK-NULLITY: $\dim(\text{NUL}(A)) + \text{RANK}(A) = 5 \quad (=N)$

$\dim(\text{NUL}(A)) = 5 - \text{RANK}(A) = 5 - 3 = 2$

(d) FIND A BASIS OF $\text{NUL}(A)$

SINCE $\dim(\text{NUL}(A)) = 2$, ENOUGH TO FIND 2 SPANNING VECTORS IN $\text{NUL}(A)$

SINCE ROW-REDUCTION PRESERVES SOLUTIONS, $Ax = 0 \iff A'x = 0$

$\text{NUL}(A) = \text{NUL}(A')$

SOLVE $A'x = 0 \Rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$$\begin{cases} x_1 - 2x_2 - 4x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_5 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_2 + 4x_4 \\ x_3 = -2x_4 \\ x_5 = 0 \end{cases}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + 4x_4 \\ x_2 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

SO $\text{NUL}(A) = \text{NUL}(A') = \text{SPAN}\underbrace{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}}_{\text{2 VECTORS}}$

BASIS $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

NOTE
1) NON-PIVOT COLS CORRESPOND TO FREE VARIABLES (HERE: x_2, x_4)
2) $\dim(\text{NUL}(A)) = \#$ FREE VARS
3) PIVOT IN EVERY COLUMN OF $A \Rightarrow \text{NUL}(A) = \{\underline{0}\}$ (B/C NO FREE VARS)

II - EXISTENCE OF SOLUTIONS

(NOW THAT WE KNOW HOW TO FIND SOLUTIONS OF EQUATIONS, OUR NEXT Q IS: DO SOLUTIONS EVEN EXIST? B/C IF THEY DON'T, WE'RE IN TROUBLE)

LUCKILY, THERE ARE VERY NEAT TESTS TO FIGURE OUT IF WE HAVE A SOL OR NOT

NOTE FACT $Ax = b$ IS CONSISTENT $\Leftrightarrow$ $b \in Col(A)$
"HAS A SOL"

EX $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} \Leftrightarrow x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in Span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\Leftrightarrow b \in Col(A)$

THEOREM [RANK CRITERION]

$Ax = b$ IS CONSISTENT $\Leftrightarrow$ RANK $[A \mid b]$ = RANK (A)

EX $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b}$

$\text{RANK}(A) = \text{RANK}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) \overset{RREF}{=} \text{RANK}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1$

BUT $\text{RANK}([A \mid b]) = \text{RANK}\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right) = \text{RANK}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 2$

SO $Ax = b$ IS INCONSISTENT

PROOF $(\Rightarrow)$ SUPPOSE $A = [v_1 \mid \cdots \mid v_n]$

THEN $Col(A) = Span\{v_1, \ldots, v_n\}$

SINCE $Ax = b$ IS CONSISTENT, $b \in \text{Col}(A)$

SO $\text{Col}([A \mid b]) = \text{Span}\{v_1, \dots, v_n, b\}$ $\quad [A \mid b] = [v_1 | \cdots | v_n | b]$

$= \text{Span}\{v_1, \dots, v_n\}$ $\quad (b \in \text{Col}(A))$

$= \text{Col}(A)$

SO $\text{rank}([A \mid b]) = \dim(\text{Col}([A \mid b])) = \dim(\text{Col}(A)) = \text{rank}(A)$

$(\Leftarrow)$ SIMILAR

IMPORTANT CONSEQUENCES

(1) $Ax = b$ IS INCONSISTENT $\Leftrightarrow$ LAST COL OF $[A \mid b]$ IS A PIVOT COL

EX
$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \; \underset{b}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$$
PIVOT COL, SO $Ax = b$ IS INCONSISTENT

WHY? SEE HW #8

(2) $Ax = b$ IS CONSISTENT FOR EVERY b $\Leftrightarrow$ A HAS A PIVOT IN EVERY ROW

EX
$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

PIVOT IN EVERY ROW, SO $Ax = b$ IS CONSISTENT FOR ALL b.

WHY? ($\Rightarrow$) IF A ($M \times N$) HAS A PIVOT IN EVERY ROW,

THEN RANK$(A) = M$

$$\begin{array}{cc} & \begin{array}{cc} N & 1 \end{array} \\ M & \left[\begin{array}{cc|c} * & & \\ & * & \end{array}\right] \\ & \begin{array}{cc} A & b \end{array} \end{array}$$

BUT THEN $[A \mid b]$ ($M \times (N+1)$) ALSO HAS A PIVOT IN EVERY ROW,

SO RANK $[A \mid b] = M$ = RANK(A)

SO $Ax = b$ IS ALWAYS CONSISTENT ✓

($\Leftarrow$) IF $Ax = b$ IS CONSISTENT FOR ALL $b \in \mathbb{F}^M$,

THEN $\text{COL}(A) = \mathbb{F}^M$

SO RANK$(A) = \text{DIM}(\text{COL}(A)) = \text{DIM}(\mathbb{F}^M) = M$

SO # PIVOTS OF $A = M$ = # ROWS OF A

SO A HAS A PIVOT IN EVERY ROW