LECTURE 22 - SYSTEMS OF LINEAR EQUATIONS (III) (SECTIONS 2.3.2)

WELCOME TO OUR FINAL LECTURE ON SYSTEMS OF EQUATIONS! TODAY WE'LL DISCUSS SOME MISCELLANEOUS TOPICS RELATED TO SYSTEMS, STARTING WITH A USEFUL DESCRIPTION OF SOLUTIONS:

I - HOMOGENEOUS AND PARTICULAR SOLUTIONS

**Theorem** The general solution of \( Ax = \mathbf{b} \) is of the form

\[
    x = x_0 + x_p
\]

where

\[
    x_0 = \text{general solution of } Ax = 0 = \text{null}(A)
\]

\[
    x_p = \text{particular solution of } Ax = b
\]

**Example** Solution of

\[
    \begin{bmatrix}
        1 & 1 \\
        1 & 1
    \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]

is

\[
    x = t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
    A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad Ax = 0 \Rightarrow \text{one sol.}
\]

of \( Ax = b \)

**Why?** 1) If \( x = x_0 + x_p \), then

\[
    Ax = A(x_0 + x_p) = Ax_0 + Ax_p = 0
\]

2) Convenient, if \( x \) solves \( Ax = b \), let \( y = x - x_p \)

Then

\[
    Ay = A(x - x_p) = Ax - Ax_p = A(x_0 + x_p) - Ax_p
\]

so \( y \) solves \( Ay = 0 \), so \( y = x_0 \) for some \( x_0 \in \text{null}(A) \)

Thus

\[
    x = y + x_p = x_0 + x_p
\]
II. Some Remarks / Consequences

(Never use this theorem to solve $Ax = b$, it's more useful for theory)

1) Geometric description of $Ax = b$

Theorem says: Solutions of $Ax = b$ are just translations of Null($A$)

$A\mathbf{x} = \mathbf{b}$

\[ x = x_0 + x_p \]

Null($A$)

$A\mathbf{x} = 0$

So geometrically, all the $Ax = b$ look the same as $Ax = 0$.

This is why Null($A$) is so important, it "controls" all the solutions.

2) Can use this to show $Ax = b$ can only have 0, 1, or many solutions.

Exactly one sol

3) Fact: If $A$ is square and $Ax = b$ has a solution for some $b$,

Then $Ax = b$ has a sol for all $b$.

Exactly one

Why? In this case, we must have Null($A$) = \{0\} because otherwise $Ax = 0$ would have 0 on infinitely many solutions

$x = \text{Null}(A) + x_p$

But Null($A$) = \{0\} \Rightarrow Null($A$) = \{0\}

\[ L = A - I \Rightarrow A \text{ is invertible} \]
Hence for any \( \mathbf{b} \), \( \mathbf{Ax} = \mathbf{b} \) has a unique solution \( \mathbf{x} = \mathbf{A}^{-1} \mathbf{b} \).

9) A similar result holds for differential equations (see HNLP).

III - BASIS

(Back to practical things! The nice thing about row-reduction is that it simplifies tasks that used to be tedious.)

**EX** Find a subset of \( S \) that is a basis for span \( (S) \), where

\[
S = \left\{ (2, -3, 5), (2, -1, 2), (1, 0, -2), (0, 2, -1), (7, 2, 0) \right\}
\]

(Before eliminating LD vectors, but now much easier!)

Find a basis for \( \text{col}(A) \), where:

\[
A = \begin{bmatrix}
2 & 0 & 1 & 0 & 2 \\
-3 & -2 & 0 & 2 & 2 \\
5 & 2 & 0 & -2 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \rightarrow \quad A' = \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Ans: \( \left\{ (2, -3, 5), (1, 0, -2), (0, 2, -1) \right\} \) (That was easy!)

III - SUMMARY OF NIEF

In fact, let me use the previous EX to "prove" you of some facts about NIEF.

**FACTS** If \( \mathbf{A} \sim \mathbf{A}' \) and \( \mathbf{A}' \) is in NIEF, then:

1) \( \mathbf{A}' \) has \( r \) nonzero rows, \( r = \text{rank}(\mathbf{A}) \)

Here: \( r \) nonzero rows

Why? \( \text{rank}(\mathbf{A}) = \# \text{ pivots} = \# \text{ pivot rows} \)
2) ONE COL OF $A'$ IS $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, ANOTHER ONE $e_2$, AND SO ON UNTIL $e_k$, $k = \text{RANK}(A)$

HERE: $e_1 = \text{COL}(1)$, $e_2 = \text{COL}(3)$, $e_3 = \text{COL}(4)$...

3) THE PIVOT COLS OF $A$ ARE LI (WHY? SEE LAST TIME)

HERE: COLS 1, 3, 4 OF $A$ ARE LI

4) THE LIN DEP RELATIONS OF THE NON-PIVOT COLS ARE PRESERVED

**EX** For $A'$, \[
\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so for } A, \begin{bmatrix} \frac{8}{2} \\ \frac{2}{2} \end{bmatrix} = 4 \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \end{bmatrix}
\]

This last fact is super useful to reconstruct $A$ from $E$.

**EX** Suppose $A' = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is INVERSE of $A$

Find $A$ if cols 1, 3, 4 of $A$ are \[
\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 22 \\ 28 \\ 33 \end{bmatrix}
\]

**Ans** $A = \begin{bmatrix} 1 & 2 & 4 & 7 & 22 \\ 2 & 4 & 5 & 8 & 28 \\ 3 & 6 & 6 & 9 & 33 \end{bmatrix}$

2 COL(1) 2 COL(1) + 7 COL(4)

**Notice**\[
\begin{bmatrix} \frac{2}{3} \\ \frac{3}{3} \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ so } 2 \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{22}{3} \\ \frac{28}{3} \end{bmatrix}
\]

COL(5) = 2 COL(1) + 7 COL(4)
**T - Basis Extension**

(LASTLY, WE CAN USE THIS IDEA TO EXTEND A LT SUBSET TO A BASIS.
WE ALREADY KNOW THAT WE CAN DO THIS IN THEORY, BUT NOW WE CAN DO IT IN PRACTICE)

**EX**  
\[ S = \{ (-2,0,0,1), (1,1,-2,-1) \} \] IS A LT SUBSET OF \( V = \mathbb{R}^4 \)

**EXTEND S TO A BASIS OF \( \mathbb{R}^4 \)**

1) **Pick any \( u \in V \) of \( V \), say**

\[ u = \{ (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1) \} \]

2) **Consider**

\[ \begin{bmatrix} u \mid \beta \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ \end{bmatrix} \]

\[ A = \begin{bmatrix} 5 \\ \beta \end{bmatrix} \]

\[ A = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1/2 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix} \]

**Ans**  
\[ \{ (-2,0,0,1), (1,1,-2,-1), (1,0,0,0), (0,1,0,0) \} \] BASIS OF \( V = \mathbb{R}^4 \)

**Note**  
For general \( V \subseteq \mathbb{F}^n \), first find a basis \( \beta \) of \( V \) and use the same trick with \( A = [ S \mid \beta] \) (See 3.4).

**Why Works**  
The first two columns of \( A \) must be pivot columns,  
so we get a linear dependence in \( S \) (by 4), so get \( \beta \) \( S \) containing those pivot cols, that is containing \( S \).