MONDAY, MAY 20, 2019

LECTURE 22 – SYSTEMS OF LINEAR EQUATIONS (III) (SECTIONS 3.3 & 3.4)

WELCOME TO OUR FINAL LECTURE ON SYSTEMS OF EQUATIONS! TODAY WE'LL DISCUSS SOME MISCELLANEOUS TOPICS RELATED TO SYSTEMS, STARTING WITH A USEFUL DESCRIPTION OF SOLUTIONS:

I – HOMOGENEOUS AND PARTICULAR SOLUTIONS

THEOREM THE GENERAL SOLUTION OF $Ax = b$ IS OF THE FORM

$$x = x_0 + x_p \quad \text{WHERE}$$

x_0 = GENERAL SOLUTION OF $Ax = \underline{0}$ = $NUL(A)$

x_p = PARTICULAR SOLUTION OF $Ax = b$

EX SOLUTION OF $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 3 \\ 3 \end{bmatrix}}_{b}$ IS $x = \overset{(x_0)}{\underbrace{t\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{Ax=\underline{0}}} + \overset{(x_p)}{\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{ONE SOL OF } Ax=b}}$

WHY? 1) IF $x = x_0 + x_p$, THEN

$$Ax = A(x_0 + x_p) = \underbrace{Ax_0}_{0} + \underbrace{Ax_p}_{b} = b$$

2) CONVERSELY, IF x SOLVES $Ax = b$, LET $y = x - x_p$

THEN $Ay = A(x - x_p) = \underbrace{Ax}_{b} - \underbrace{Ax_p}_{b} = b - b = 0$

SO y SOLVES $Ay = 0$, SO $y = x_0$ FOR SOME $x_0 \in NUL(A)$

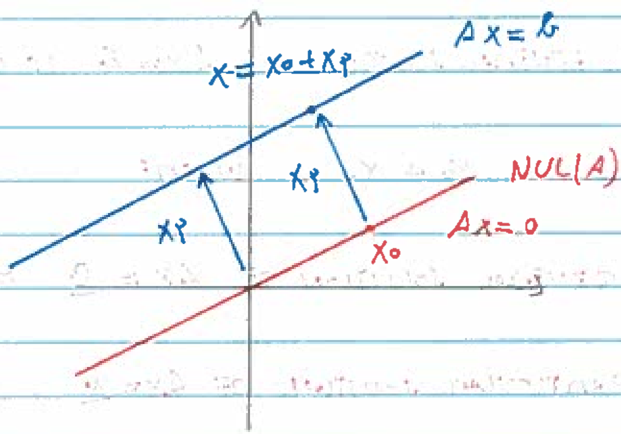
THEN $x = y + x_p = x_0 + x_p$ ■

II – SOME ~~[illegible]~~ REMARKS / CONSEQUENCES

(NEVER USE THIS THEOREM TO SOLVE $Ax = b$, IT'S MORE USEFUL FOR THEORY)

1) GEOMETRIC DESCRIPTION OF $Ax = b$

THEOREM SAYS SOLUTIONS OF $Ax = b$ ARE JUST TRANSLATES OF $NUL(A)$



(SO GEOMETRICALLY, ALL THE $Ax = b$ LOOK THE SAME AS $Ax = 0$,
THIS IS WHY $NUL(A)$ IS SO IMPORTANT, IT "CONTROLS" ALL THE SOL

2) CAN WE USE THIS TO SHOW $Ax = b$ CAN ONLY HAVE 0, 1, OR ∞ MANY SOL

3) FACT IF A IS SQUARE AND $Ax = b$ HAS ~~[illegible]~~ EXACTLY ONE SOL FOR SOME b,
THEN $Ax = b$ HAS ~~A~~ EXACTLY ONE SOL FOR ALL b

WHY? IN THIS CASE WE MUST HAVE $NUL(A) = \{0\}$ BECAUSE OTHERWISE
$Ax = b$ WOULD HAVE 0 OR INFINITELY MANY SOLUTIONS
$(x = NUL(A) + x_p)$

BUT $NUL(A) = \{0\} \Rightarrow N(L_A) = \{0\}$

$\Rightarrow L_A$ 1-1 $\Rightarrow L_A$ INV (SINCE A IS SQUARE

$\Rightarrow A$ INVERTIBLE

HENCE FOR ANY b, $Ax=b$ HAS A UNIQUE SOL; $x=A^{-1}b$ ∎

4) A SIMILAR RESULT HOLDS FOR DIFFERENTIAL EQUATIONS (SEE HW #8)

III – BASIS

(BACK TO PRACTICAL THINGS! THE NICE THING ABOUT ROW-REDUCTION IS THAT IT SIMPLIFIES TASKS THAT USED TO BE TEDIOUS)

EX FIND A SUBSET OF S THAT IS A BASIS FOR SPAN (S), WHERE

$$S=\{(2,-3,5),(8,-12,20),(1,0,-2),(0,2,-1),(7,2,0)\}$$

(BEFORE ELIMINATED LD VECTORS, BUT NOW MUCH EASIER!)

FIND A BASIS FOR COL(A), WHERE:

$$A=\begin{array}{c}\begin{bmatrix} 2 & 8 & 1 & 0 & 7 \\ -3 & -12 & 0 & 2 & 2 \\ 5 & 20 & -2 & -1 & 0 \end{bmatrix} \\ \begin{matrix}\uparrow & & \uparrow & \uparrow & \end{matrix}\end{array} \longrightarrow A'=\begin{array}{c}\begin{bmatrix} 1 & 4 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \\ \begin{matrix}\uparrow & & \uparrow & \uparrow & \end{matrix}\end{array}$$

ANS $\{(2,-3,5),(1,0,-2),(0,2,-1)\}$ (THAT WAS EASY!)

IV – SUMMARY OF RREF

IN FACT, LET ME USE THE PREVIOUS EX TO "REMIND" YOU OF SOME FACTS ABOUT RREF:

FACTS IF $A \sim A'$ AND A' IS IN RREF, THEN:

1) A' HAS K NONZERO ROWS, $K=\text{RANK}(A)$

HERE: 3 NONZERO ROWS

WHY? RANK(A) = # PIVOTS = # PIVOT ROWS

2) ONE COL OF A' IS $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ANOTHER ONE e_2,

AND SO ON UNTIL e_k, $k = \text{RANK}(A)$

HERE: $e_1 = \text{COL}(1)$, $e_2 = \text{COL}(3)$, $e_3 = \text{COL}(4)$

3) THE PIVOT COLS OF A ARE LI (WHY? SEE LAST TIME)

HENCE COLS 1, 3, 4 OF A ARE LI

4) THE LIN DEP RELATIONS OF THE NON-PIVOT COLUMNS ARE PRESERVED

EX FOR A', $\underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}_{\text{COL}(2)} = 4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{COL}(1)}$, SO FOR A, $\underbrace{\begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}}_{\text{COL}(2)} = 4 \underbrace{\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}}_{\text{COL}(1)}$

THIS LAST FACT IS SUPER USEFUL TO RECONSTRUCT A FROM RREF

EX SUPPOSE $A' = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ IS RREF OF A

FIND A IF COLS 1, 3, 4 OF A ARE $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

ANS $A = \begin{bmatrix} 1 & 2 & 4 & 7 & 22 \\ 2 & 4 & 5 & 8 & 28 \\ 3 & 6 & 6 & 9 & 33 \end{bmatrix}$

$2\,\text{COL}(1)$ $\quad$ $2\,\text{COL}(1) + 3\,\text{COL}(4)$

NOTICE: $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, SO $2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 22 \\ 28 \\ 33 \end{bmatrix}$

$\text{COL}(5) = 2\,\text{COL}(1) + 3\,\text{COL}(4)$

V – BASIS EXTENSION

(LASTLY, WE CAN USE THIS IDEA TO EXTEND A LI SUBSET TO A BASIS. WE ALREADY KNOW THAT WE CAN DO THIS IN THEORY, BUT NOW WE CAN DO IT IN PRACTICE)

EX $S = \{(-2,0,0,1), (1,1,-2,-1)\}$ IS A LI SUBSET OF $V = \mathbb{R}^4$

EXTEND S TO A BASIS OF $\mathbb{R}^4$

COOL TRICK 1) PICK ANY BASIS β OF V, SAY

$$\beta = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

2) CONSIDER

$$A = [S \mid \beta] = \begin{array}{c} \begin{array}{cccccc} \downarrow & \downarrow & \downarrow & \downarrow & & \end{array} \\ \left[\begin{array}{cccccc} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\underbrace{}_{S} \qquad \underbrace{}_{\beta}$$

$$\longrightarrow A' = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & -1/2 & 1 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 2 \\ 0 & 0 & 0 & 1 & 1/2 & 0 \end{array}\right]$$

S

ANS $\{\underbrace{(-2,0,0,1), (1,1,-2,-1)}_{S}, (1,0,0,0), (0,1,0,0)\}$ BASIS OF $V = \mathbb{R}^4$

NOTE FOR GENERAL $V \subseteq \mathbb{F}^N$, FIRST FIND A BASIS β OF V AND USE THE SAME TRICK WITH $A = [S \mid \beta]$ (SEE 3.4).

WHY WORKS THE FIRST TWO COLUMNS OF A MUST BE PIVOT COLUMNS, O/W GET A LIN DEP RELATION IN S (BY 4)), SO GET A BASIS CONTAINING THOSE PIVOT COLS, THAT IS CONTAINING S