Math 110, Professor Ogus, Homework due 2/28
(written by Janak Ramakrishnan)

3.3.1. (a) F, (b) F, (c) T, (d) F, (e) F, (f) F, (g) T, (h) F.

3.3.2. **d.** Adding the first two equations together, we note that $3x_1 = 0$, so $x_1 = 0$. Then from the second equation, $x_2 = x_3$. Thus, a basis is $(0, 1, 1)$ and the dimension is 1.

g. These two equations are clearly linearly independent (one has x_1 and the other does not). Subtracting the second equation from the first, we get $x_1 + x_2 + 2x_3 = 0$. Then the dimension is 2, since we have 2 equations and 4 unknowns, and $(2, 0, -1, -1)$ and $(0, 2, -1, -3)$ are a basis.

3.3.3. **d.** Repeating the above argument, we see that for one solution, $x_1 = 2$, and so $x_3 - x_2 = -1$, $2(x_2 - x_3) = 2$, so $x_2 - x_3 = 1$, and thus $x_3 = 0$, $x_2 = 1$. Therefore, the set of all solutions is $\{(x_1, x_2, x_3) \in F^3 \mid x_1 = 2, x_2 = 1 + t, x_3 = t, t \in F\}$, where F is our base field.

g. Again subtracting the second equation from the first, we have $x_1+x_2+2x_3 = 0$. Thus, $(2, 0, -1, 0)$ is a solution. Therefore, the set of all solutions is $\{(x_1, x_2, x_3, x_4) \in F^4 \mid x_1 = 2t+2, x_2 = 2s, x_3 = -1 - t - s, x_4 = -t - 3s, t, s \in F\}$.

3.3.6. We have the equations $a + b = 1$ and $2a - c = 11$. The easiest way to solve these is to make $a = 0$, $b = 1$, and $c = -11$. The set of all solutions to $T(\vec{v}) = \vec{0}$ is spanned by $(1, -1, 2)$. Thus $T^{-1}(1, 11) = \{(a, b, c) \mid a = t, b = 1 - t, c = -11 + 2t, t \in \mathbb{R}\}$.

3.3.8. Note that $R(T)$ has dimension 2, since $(1, 0, 1)$ and $(1, 1, 0)$ are in it, but if $(x, y, z) \in R(T)$, then $x = y + z$, so in particular $R(T) \neq \mathbb{R}^3$. Since $\{(x, y, z) \in \mathbb{R}^3 \mid x = y + z\}$ has dimension 2 and contains $R(T)$, it must be $R(T)$, so we need only check this equation. $1 \neq 3 + 2$, so $(1, 3, 2) \notin R(T)$, but $2 = 1 + 1$, so $(2, 1, 1) \in R(T)$.

~~**3.4.1.** (a) F, (b) T, (c) T, (d) T, (e) F, (f) T, (g) T.~~

3.4.2d. We form the augmented matrix and solve:

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 2 & -1 & 6 & 6 & -2 \\ -2 & 1 & -4 & -3 & 0 \\ 3 & -2 & 9 & 10 & -5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & -1 & -8 & 3 & -14 \\ 0 & 1 & 15 & 1 & 16 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 5 & 1 & 4 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 1.5 & -1 \\ 0 & 0 & 5 & 1 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 3/2 & -1 \\ 0 & 0 & 0 & -13/2 & 9 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 3/2 & -1 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 0 & 14/13 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 0 & 0 & 16/13 \\ 0 & 0 & 1 & 0 & 14/13 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7/13 \\ 0 & 1 & 0 & 0 & 16/13 \\ 0 & 0 & 1 & 0 & 14/13 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right)$$

Thus, the solution is $(7/13, 16/13, 14/13, -18/13)$.

3.4.5. Let the rows of the RREF of A be v_1, v_2, v_3. Since each row of A is a linear combination of the rows in the RREF of A, we know that the first row of A must equal $v_1 + v_3$. Thus, the third coordinate is 2. The second row of A must be $-v_1 - v_2 - 2v_3$, so the third coordinate is $-2 + 5 = 3$. Finally, the third row of A must be $3v_1 + v_2$, so the third coordinate is 6.

3.4.13. Let $v_1 = (1, 0, 1, 1, 1, 0)$ and $v_2 = (0, 2, 1, 1, 0, 0)$, so $S = \{v_1, v_2\}$. For part a, it is clear that v_1 and v_2 are linearly independent (v_1 has a nonzero first coordinate). Since $1 - 0 + 0 \cdot 1 + 2 \cdot 1 - 3 \cdot 1 + 0 = 0$ and $2 \cdot 1 - 0 - 1 + 3 \cdot 1 - 4 \cdot 1 + 4 \cdot 0 = 0$, $v_1 \in V$, and likewise, $2 \cdot 0 - 2 + 0 \cdot 1 + 2 \cdot 1 - 3 \cdot 0 + 0 = 0$ and $2 \cdot 0 - 2 + -1 + 3 \cdot 1 - 4 \cdot 0 + 4 \cdot 0 = 0$ show that $v_2 \in V$. It is easy to verify that V is 4-dimensional. The vectors $v_3 = (0, 0, -5, 1, 0, -2)$ and $v_4 = (0, 0, 1, 3, 2, 0)$ complete S to a basis – it is easy to check that they lie in V, so we need only check that $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set. But v_3 is clearly independent from $\{v_1, v_2\}$, since its last coordinate is nonzero, and both v_1 and v_2 have zero last coordinate. v_4 is independent from v_1, v_2, v_3 because any linear combination of v_1, v_2, v_3 with nonzero coefficient for v_1 must have nonzero first coordinate, and likewise for v_2. Thus, we need only consider multiples of v_3, but v_3 and v_4 are easily linearly independent.

3.4.15. Let B and B' be two matrices in RREF, with B,B' equal to A in RREF. We know from the theorem that B and B' are $m \times n$, with r nonzero rows, where $r = \text{Rank}(A)$, or else there is no way for B' to be the RREF of A. We will show $B = B'$.

First we show that for each $i \leq n$, if we take the first j such that either $b_j = e_i$ or $b'_j = e_i$, then $b_j = b'_j = e_i$. We show this by contradiction, so assume that it is not true for some i. We can choose the least such i. WLOG, we can assume that $b_j = e_i$ – else we can switch B and B'. For B' to be in RREF, since e_i is not in the first j columns of B', b'_j cannot have any nonzero component of $e_i, \ldots, e_r$. Then it is easy to see that b_j is a linear combination of $e_1, \ldots, e_{i-1}$, say $d'_1 e_1 + \ldots + d'_{i-1} e_{i-1}$. If we let j_k (for $k < i$) be the first column at which $b_{j_k} = b'_{j_k} = e_k$, then by Theorem 3.16(d), this means that $a_j = d'_1 a_{j_1} + \ldots d'_{j_{i-1}} a_{j_{i-1}}$. But now consider B and Theorem 3.16(c): we can let $j_i = j$, and so we know that a_j is linearly independent from $a_{j_1}, \ldots, a_{j_{i-1}}$, which is a contradiction. Thus, there is no such i – for each i and the first j such that $b_j = e_j$, $b'_j = e_j$ as well.

Thus, we can choose $j_1, \ldots, j_r$ as in Theorem 3.16 to be the same for B and B'. Now we note that the converse of Theorem 3.16(d) is also true, since there is only one possible representation of each column of A in terms of $a_{j_1}, \ldots, a_{j_r}$, as they are linearly independent, by Theorem 3.16(c). Thus, column k of A determines column k of B and of B', showing that $B = B'$.