2.3.9 Define $U(x,y) = (x,0)$ and $T(x,y) = (0, x+y)$. Then $UT(a,b) = U(0, a+b) = (0,0)$ for any $(a,b)$, but $TU(a,b) = T(a,0) = (a,0) \neq 0_V$ for $a \neq 0$. By taking the matrix representations of $U$ and $T$ with respect to the standard basis we get $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ respectively. One can check that these matrices satisfy the desired properties.

2.3.11 Let $V$ be a vector space and $T : V \to V$ be linear. Prove $T^2 = T_0 \iff R(T) \subseteq N(T)$.

$\Rightarrow$ Take $v \in V$ and consider $T^2(v) = T(T(v))$. Since $T(v) \in R(T)$, by assumption, $T(v) \in N(T)$. Therefore $T^2(v) = T(T(v)) = 0$ for all $v \in V$.

$\Leftarrow$ Take $w in R(T)$. By definition of the range $w = T(v)$ for some $v \in V$. Then $T(w) = T(T(v)) = T^2(v) = T_0(v) = 0$, so $w \in N(T)$. Thus $R(T) \subseteq N(T)$.

2.3.15 Let $M$ be an $m \times n$ matrix, and $A$ be a $n \times p$ matrix. Let $A_j$ denote the $j^{th}$ column of $A$ and similarly for $(MA)_j$. Assume $A_j = c_1 A_1 + \cdots + c_n A_n$ (where $c_j = 0$) for some $j$. Thus we have $A_{(k,j)} = c_1 A_{(k,1)} + \cdots + c_n A_{(k,n)}$ for all $k$. By the formula for matrix multiplcation,we have

$$(MA)_{(i,j)} = \sum_{k=1}^{n} (M_{(i,k)} A_{(k,j)}) = \sum_{k=1}^{n} (M_{(i_k)} (c_1 A_{(k,1)} + \cdots + c_n A_{(k,n)})$$

$$= \sum_{l=1}^{n} c_l \sum_{k=1}^{n} M_{(i,k)} A_{(k,l)} = \sum_{l=1}^{n} c_l (MA)_{(i,l)}.$$

Thus, the $j^{th}$ column of $MA$ is also a linear combination of the other columns with the same corresponding coefficients.

2.4.1 (a) False.
(b) True.
(c) False.
(d) False.
(e) True.
(f) False.
(g) True.
(h) True.
(i) True.

2.4.2 (a) T is not invertible. The dimension of the codomain is larger than the dimension of the domain, so T cannot be onto.

(c) T is invertible. One can check that T is both one-to-one and onto.

(f) T is invertible. One can check that T is both one-to-one and onto.

2.4.5 Let $A$ be invertible. Prove that $A^t$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.
*Proof:* If $A$ is invertible then $(A^{-1})^t$ exists. Consider $A^t(A^{-1})^t = ((A^{-1})A)^t = I_n^t = I_n$. Similarly, $(A^{-1})^t A^t = (A(A^{-1}))^t = I_n^t = I_n$. Therefore we get the desired result.

2.4.10 (a) By exercise (9), if $A$ and $B$ are $n \times n$ and $AB$ is invertible, then $A$ and $B$ are invertible. In this situation, $AB = I_n$ which is an invertible matrix. Therefore $A$ and $B$ are invertible.

(b) Since $B$ is invertible, the $B^{-1}$ exists. By multiplying both sides of the equation on the right by $B^{-1}$ we get $(AB)B^{-1} = A(BB^{-1}) = A(I_n) = A = I_n(B^{-1}) = B^{-1}$.

(c) Let $V, W$ be $n$ dimensional vector spaces, $T : V \to W$, $U : W \to V$ linear transformations. If $UT = I_V$, then $T$ and $U$ are invertible.
*Proof:* If $UT = I_V$, then $R(U) = V$ and $U$ is onto. Thus, by the Rank-Nullity Theorem, we see that $U$ is one-to-one. Therefore $U$ is invertible. By the same argument as in (b), we can show that $U^{-1} = T$, so $T$ in invertible as well.