Math 121A – Homework 6

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Reading: Sections 2.5 and 2.6. Since you're probably exhausted from the exam, this assignment is shorter than usual. There will be more 2.6 problems on HW 7.

Note: Section 2.5 in the book is really badly written, so I *exceptionally* allow you not to read that section. If you understand what I did in lecture and are able to do the homework problems, you should be 100 % fine. Also, my notation (although standard) is different from the one of the book. In order to make the homework problems less confusing, I rewrote the book-problems to make it more consistent with the notation in lecture. The answers to the AP problems (except AP4) should match the answers of the corresponding problems in the back of the book.

- **Section 2.5:** AP1, AP2, AP3, 7, 10, AP4 (Optional: AP5, AP6, AP7)
- **Section 2.6:** 2(b)(d)(e), 3(a), 19 (only do the finite-dimensional case)

AP 1 (Modification of 2.5.1) Label the following statements as True or False

(a) Suppose $\beta = \{v_1, \cdots, v_n\}$ and $\gamma = \{w_1, \cdots, w_n\}$ are ordered bases of a vector space V and Q is the change of coordinates matrix that changes β coordinates into γ coordinates. Then the j-th column of Q is $[w_j]_\beta$

(b) Every change of coordinates matrix is invertible

(c) Let T be a linear operator on a finite-dimensional vector space V, let β and γ be ordered bases for V and let Q be the change of coordinates matrix that changes β-coordinates to γ coordinates. Then $[T]_\gamma^\gamma = Q[T]_\beta^\beta Q^{-1}$

(d) Two matrices $A, B \in M_{n\times n}(F)$ are called similar if $A = Q^T BQ$ for some $Q \in M_{n\times n}(F)$

(e) Let T be a linear operator on a finite-dimensional vector space V, then for any ordered bases β and γ of V, $[T]_\gamma^\gamma$ is similar to $[T]_\beta^\beta$

AP 2 (Modification of 2.5.2(b)) Let $V = \mathbb{R}^2$. Find the change of coordinates matrix that changes $\beta-$coordinates to $\gamma-$coordinates, where

$$\beta = \{(0, 10), (5, 0)\}$$

$$\gamma = \{(-1, 3), (2, -1)\}$$

AP 3 (Modification of 2.5.4) Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T(a, b) = (2a + b, a - 3b)$$

Let β be the standard basis of $\mathbb{R}^2$ and let

$$\gamma = \{(1, 1), (1, 2)\}$$

Calculate $[T]_\beta^\beta$ and use this to find $[T]_\gamma^\gamma$

AP 4 If V is finite-dimensional and $T : V \to V$ is linear then the trace of T is defined as

$$tr(T) = tr([T]_\beta^\beta)$$

where β is any basis of V.

(a) Use Problem 10 in 2.5 to show that the trace of T is well-defined, meaning if γ is any other basis of V, then

$$tr([T]_\gamma^\gamma) = tr([T]_\beta^\beta)$$

(b) Let $V = P_2(F)$ and define $T : V \to V$ by $T(p(x)) = p'(x)$. Calculate $tr(T)$

Hint: Choose a 'smart' basis of V and use the definition of $tr(T)$

Optional AP 5 (Modification of 2.5.3(d)) Let $V = P_2(\mathbb{R})$ Find the change of coordinates matrix that changes $\beta-$coordinates to $\gamma-$coordinates, where

$$\beta = \left\{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\right\}, \gamma = \left\{x^2 - x + 1, x + 1, x^2 + 1\right\}$$

Optional AP 6 (Modification of 2.5.5) Let $V = P_1(\mathbb{R})$ and let T be the linear operator on V defined by $T(p(x)) = p'(x)$. Let $\beta = \{1, x\}$, $\gamma = \{1 + x, 1 - x\}$. Find $[T]_\beta^\beta$ and use this to find $[T]_\gamma^\gamma$.

Optional AP 7 (Modification of 2.5.6(a)) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and let $\gamma = \{(1, 1), (1, 2)\}$. Find $[L_A]_\gamma^\gamma$ and find an invertible matrix Q with $[L_A]_\gamma^\gamma = QAQ^{-1}$