Math 121A – Homework 7

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Reading: Sections 2.6, 3.1, 3.2. Some of the optional problems in 2.6 explore applications of dual spaces in pure (Problems 11, $14-16$, and AP) and applied math (Problem 10, really cool) you should check them out, they're really fun! There will be YouTube videos about them at some point. Also in 3.2, ignore the proof of Theorem 3.6, but understand Example 3.

- **Section 2.6:** 1, 6, 13(a), 20(a) (Optional: 7, 14, 15, 16, 11, 10, AP)
- **Section 3.1:** 1, 3, 8, 9
- **Section 3.2:** 1, 4(a), 5(a)(f), 6(a)(d), 7, 14

Hint for 20(a): First of all, you're allowed to use without proof the result of 19. Show that T is not onto if and only if T^T is not one-to-one.

$(\Rightarrow)$ Suppose T is not onto, and let $W' = T(V)$, which is a proper subspace of W and use the result of 19 to find $\mathrm{g} \in W^\star$ not the zero functional, and use that to conclude that T^T is not one-to-one.
$(\Leftarrow)$ Suppose T^T is not one-to-one, so there is $\mathrm{g} \in W^\star$ not the zero functional with $T^T(\mathrm{g}) =$ the zero functional in $V^\star$ and use that to conclude T is not onto.

Some solutions:

- Solution to 2.6.13(a) and 2.6.14: Annihilator
- Solution to 2.6.15 and 2.6.16: Rank(A) = Rank(A^T)

- Solution to 2.6.11: $T^{TT} = T$?
- Solution to 2.6.10: Dual Lagrange Interpolation
- Solution to AP: Dirac Delta Functional

Optional Additional Problem: If you're curious how linear algebra is used in analysis, here's a neat application.

Let $V = C[-1, 1]$, the vector space of continuous functions f from $[-1, 1]$ to $\mathbb{R}$. In this problem, I want to show you that $V^\star$ is, in some sense, much bigger than V (so they're not isomorphic)

Given $f \in V$, define the following functional $\hat{f} \in V^\star$:

$$\hat{f}(g) = \int_{-1}^{1} f(x)g(x)dx, \quad g \in C[-1, 1]$$

(a) Calculate $\hat{f}(g)$ where $f(x) = x$ and $g(x) = x^3$

(b) Show that $\hat{f} \in V^\star$, that is $\hat{f}(g)$ is linear in g

(c) Define $T : V \to V^\star$ by $T(f) = \hat{f}$. Show that T is linear and one-to-one. In particular, it follows that $\dim(V^\star) \geq \dim(V)$.
Hint: At some point you'll need to choose $g = f$

(d) (Requires Math 140A) Show that T is not onto as follows: Define $\delta \in V^\star$ by $\delta(f) = f(0)$, show that δ cannot be written as $\hat{f}$ for any f.

Hint: Suppose $\delta = \hat{f}$ for some $f \in V$. Suppose there is some $x \neq 0$ such that $f(x) \neq 0$. Without loss of generality, assume $f(x) > 0$. Because f is continuous, there is some $\epsilon > 0$ small enough such that $f > 0$ on $(x - \epsilon, x + \epsilon)$. Let g be any continuous function that is positive on that interval, but 0 anywhere else. In particular $g(0) = 0$ (if ϵ is small enough). Then use $\delta(g) = \hat{f}(g)$ to reach a contradiction. Therefore $f = 0$ except maybe at $x = 0$, but since f is continuous this implies f is the zero function and hence $\delta = \hat{f} = \hat{0} = 0$, and hence δ is the zero functional, which is a contradiction.

Discussion: δ is called the Dirac delta functional (at 0). (d) is essentially saying that if δ is of the form $\hat{f}$, then f has to be 0 everywhere, but with an infinite 'spike' at 0, which actually exactly what its graph looks like, see for example Dirac delta. That's why the Dirac delta is strictly speaking not a function, but a functional (also called a distribution).