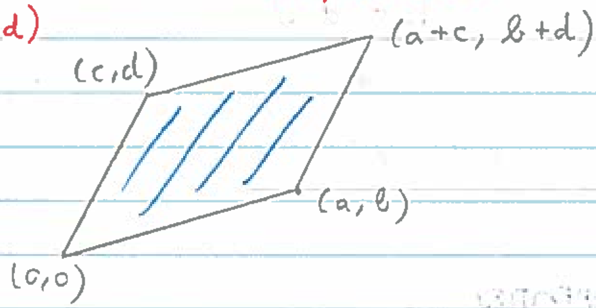
WEDNESDAY, MAY 22, 2019

LECTURE 23 – DETERMINANTS (I) (SECTIONS 4.1 & 4.2)

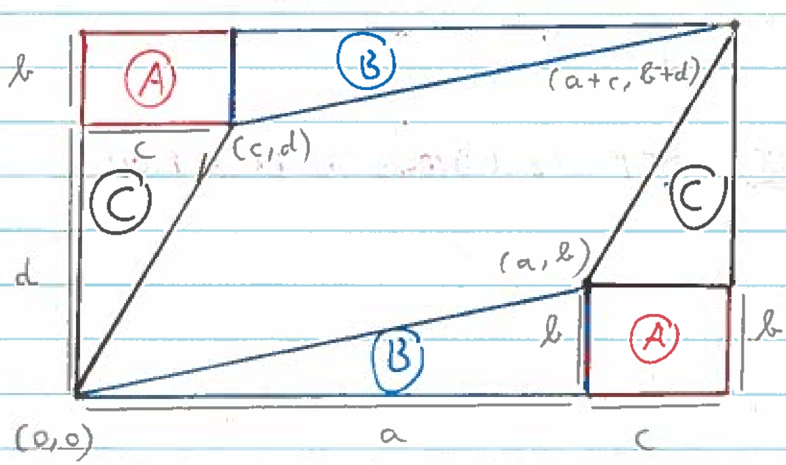
:) WELCOME TO MY FAVORITE LA TOPIC OF ALL TIME: DETERMINANTS! OVER THE NEXT 3 LECTURES, WE WILL PROVE ALL THE CUTE FACTS ABOUT DET THAT YOU LEARNED IN 3A, BUT FIRST LET ME TELL YOU WHERE THE FORMULA FOR DET COMES FROM!

I – AREA OF A PARALLELOGRAM

EX FIND THE AREA OF THE PARALLELOGRAM WITH VERTICES $(0,0)$, (a,b), (c,d), $(a+c, b+d)$ $(a>c>0,\ b>d>0)$



TRICK (KEY OBSERVATION IS THAT THE ▱ IS INSCRIBED IN A GIANT RECTANGLE)



AREA(▱) = AREA(▭) − 2(A) − 2(B) − 2(C)

$$= (a+c)(b+d) - 2bc - 2\left(\tfrac{1}{2}ab\right) - 2\left(\tfrac{1}{2}cd\right)$$

$$= \cancel{ab} + ad + bc + \cancel{cd} - 2bc - \cancel{ab} - \cancel{cd}$$

$$= ad - bc \quad (!)$$

(WHICH LEADS TO THE DEF OF THE DET:)

DEF IF $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\text{DET}(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

EX $\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1$

II − PROPERTIES

1) ⚠ DET(A) IS NOT LINEAR! $|A+B| \neq |A| + |B|$
$|cA| \neq c|A|$

EX $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$|A+B| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$ BUT $|A| + |B| = 1 + 1 = 2$

2) BUT DET IS LINEAR IN EACH ROW "MULTILINEAR"

NOTATION $\begin{vmatrix} U \\ V \end{vmatrix}$ = DET OF MATRIX WITH ROWS U & V

EX $U = (1,2)$, $V = (3,4)$, $\begin{vmatrix} U \\ V \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

FACT $\begin{vmatrix} U + KV \\ W \end{vmatrix} = \begin{vmatrix} U \\ W \end{vmatrix} + K \begin{vmatrix} V \\ W \end{vmatrix}$

FIXED → W

FIXED → $\begin{vmatrix} U \\ V+KW \end{vmatrix} = \begin{vmatrix} U \\ V \end{vmatrix} + K \begin{vmatrix} U \\ W \end{vmatrix}$

WHY? IF $U = (a, b)$, $V = (c, d)$, $W = (e, f)$

$$\begin{vmatrix} U+KV \\ W \end{vmatrix} = \begin{vmatrix} a+Kc & b+Kd \\ e & f \end{vmatrix}$$

$= (a+Kc)f - (b+Kd)e$

$= af + Kcf - be - Kde$

$= (af - be) + K(cf - de)$

$= \begin{vmatrix} a & b \\ e & f \end{vmatrix} + K \begin{vmatrix} c & d \\ e & f \end{vmatrix}$

$= \begin{vmatrix} U \\ W \end{vmatrix} + K \begin{vmatrix} V \\ W \end{vmatrix}$ ✓

3) $\begin{vmatrix} U \\ U \end{vmatrix} = 0$

4) $|I| = 1$

COOL FACT IF $f : M_{2\times 2} \to \mathbb{F}$ IS ANY FUNCTION THAT SATISFIES 2), 3), 4), THEN $f = \text{DET}$

ROW-OPS (ALL THESE PROPERTIES ARE TRUE FOR $N \times N$ DETS)

5) $\begin{vmatrix} V \\ U \end{vmatrix} = -\begin{vmatrix} U \\ V \end{vmatrix}$

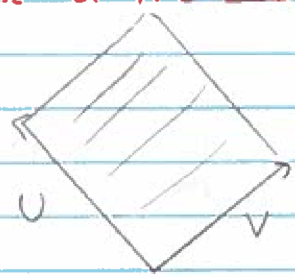
6) $\begin{vmatrix} KU \\ V \end{vmatrix} = K \begin{vmatrix} U \\ V \end{vmatrix}$, $\begin{vmatrix} U \\ KV \end{vmatrix} = K \begin{vmatrix} U \\ V \end{vmatrix}$ (Follows from 2))

7) (XK) $\begin{vmatrix} U \\ V \end{vmatrix} = \begin{vmatrix} U \\ V+KU \end{vmatrix}$

WHY? $\begin{vmatrix} U \\ V+KU \end{vmatrix} \overset{2)}{=} \begin{vmatrix} U \\ V \end{vmatrix} + K \underbrace{\begin{vmatrix} U \\ U \end{vmatrix}}_{=0} = \begin{vmatrix} U \\ V \end{vmatrix}$

III — ORIENTATION

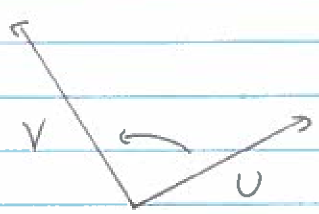
EX Find the area of the ▱ determined by $U = (-1, 5)$, $V = (4, 2)$



FACT Area $= \left| \det \begin{bmatrix} U \\ V \end{bmatrix} \right| = \left| \det \begin{bmatrix} -1 & 5 \\ 4 & 2 \end{bmatrix} \right| = |-22| = 22$

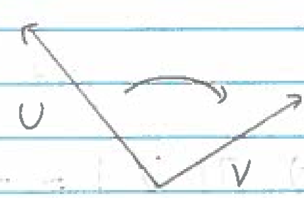
NOTE The quantity $\dfrac{\det \begin{bmatrix} U \\ V \end{bmatrix}}{\left| \det \begin{bmatrix} U \\ V \end{bmatrix} \right|} = \dfrac{-22}{22} = -1$ is called the

orientation $O\begin{bmatrix} U \\ V \end{bmatrix}$ of the ▱



$O\begin{bmatrix} U \\ V \end{bmatrix} = 1$

(Right-handed coord system)



$O\begin{bmatrix} U \\ V \end{bmatrix} = -1$

(Left-handed coord system)

IV – NXN DETERMINANTS

(NOW LET'S MOVE ON TO HIGHER-ORDER DETERMINANTS AND LET ME "REMIND" YOU HOW TO TAKE THE DET OF A 3x3 MATRIX)

$$\begin{vmatrix} 2 & 3 & 0 \\ 2 & 0 & 3 \\ -1 & 1 & 1 \end{vmatrix} = +2\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} \quad (= -21,$$

$$= (-1)^{1+1} A_{11} |\tilde{A}_{11}| + (-1)^{1+2} A_{12} |\tilde{A}_{12}| + (-1)^{1+3} A_{13} |\tilde{A}_{13}|$$

NOTATION $A_{ij} = (i,j)^{TH}$ ENTRY OF A

$\tilde{A}_{ij}$ = MATRIX OBTAINED BY DELETING ROW i & COL j OF A

DEF $$|A| = \sum_{j=1}^{N} (-1)^{1+j} A_{1j} |\tilde{A}_{1j}|$$ "COFACTOR EXPANSION ALONG ROW 1"

NOTE NICE, B/C COFACTOR EXPANSION ALONG ROW i IS:

$$\sum_{j=1}^{N} (-1)^{i+j} A_{ij} |\tilde{A}_{ij}|$$

Q DOES THAT GIVE US THE SAME ANSWER? YES

KEY LEMMA (USED NEXT TIME)

SUPPOSE ROW i OF A IS OF THE FORM e_j

$$A = \begin{array}{cc} & \begin{matrix} & & & j & & & & \\ & & & \downarrow & & & & \end{matrix} \\ i \rightarrow & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} e_j \end{array}$$

THEN $|A| = (-1)^{i+j} |\tilde{A}_{ij}|$

(PROOF IS BY INDUCTION ON SIZE OF A, AND IS ANNOYING)

NOTE THIS IS THE ANSWER YOU SHOULD GET W/ COFACTOR EXPANSION ALONG ROW i, BUT THE POINT IS THAT YOU'RE ONLY USING AN EXPANSION ALONG ROW 1.

V - MULTILINEARITY (IF TIME PERMITS)

(GOAL OF NEXT TIME IS TO GENERALIZE THE PROPERTIES OF 2×2 MATRICES FROM TODAY, STARTING W/ MULTILINEARITY)

NOTATION IF $a_1, \dots, a_N$ ARE VECTORS, THEN

$$\begin{vmatrix} a_1 \\ \vdots \\ a_N \end{vmatrix} = \text{DET OF MATRIX WITH ROWS } a_1, \dots, a_N$$

FACT FOR ALL $r = 1, \dots, N$,

$$\text{row } r \longrightarrow \begin{vmatrix} a_1 \\ \vdots \\ u + kv \\ \vdots \\ a_N \end{vmatrix} = \begin{vmatrix} a_1 \\ \vdots \\ u \\ \vdots \\ a_N \end{vmatrix} + k \begin{vmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_N \end{vmatrix}$$

(PROOF NEXT TIME)