FRIDAY, MAY 24, 2019

LECTURE 24 – DETERMINANTS (II) (SECTION 4.2)

) ALRIGHT, LET'S CONTINUE OUR EXPLORATION OF DET! SO FAR I DEFINED THE DET BY EXPANDING IT ALONG THE FIRST ROW, BUT TODAY I'LL SHOW YOU THAT YOU'LL GET THE SAME ANSWER BY EXPANDING IT ALONG ANY ROW.

I – MULTILINEARITY

IT'S ALL BASED ON THE FOLLOWING GENERALIZATION OF MULTILINEARITY)

DEF

$$\begin{vmatrix} a_1 \\ \vdots \\ a_N \end{vmatrix} = \text{DET OF MATRIX WHOSE ROWS ARE } a_1, \dots, a_N$$

FACT [MULTILINEARITY] (DET IS LINEAR IN EVERY ROW)

FOR ALL $r = 1, \dots, N$

$$\text{ROW } r \rightarrow \begin{vmatrix} a_1 \\ \vdots \\ u + kv \\ \vdots \\ a_N \end{vmatrix} = \begin{vmatrix} a_1 \\ \vdots \\ u \\ \vdots \\ a_N \end{vmatrix} + k \begin{vmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_N \end{vmatrix}$$

WHY? NOTE CAN CHECK $r = 1$ SEPARATELY (SEE HW), SO LET $r \geq 2$

MAIN IDEA INDUCTION ON N (SUPPOSE TRUE FOR $(N-1) \times (N-1)$ MATRICES)

$$|A| \overset{\text{DEF}}{=} \sum_{j=1}^{N} (-1)^{1+j} A_{1j} \left| \tilde{A}_{1j} \right|$$

$$\begin{array}{c} a_1 \\ a_2 \end{array} \begin{vmatrix} A_{11} & A_{1j} & A_{1N} \\ A_{21} & A_{2j} & A_{2N} \end{vmatrix}$$

(POINT: IN $\tilde{A}_{1j}$ YOU DELETE ROW 1 AND j^{TH} ENTRY OF OTHER ROWS)

NOTATION: FOR FIXED j, $\hat{a_i} = (a_i$ WITHOUT j^{TH} ENTRY$)$

EX $j = 3$, $a_i = (2, 4, \cancel{6}, 8)$, $\hat{a_i} = (2, 4, 8)$

THEN $$|A| = \sum_{j=1}^{N} (-1)^{1+j} A_{1j} \underbrace{\begin{vmatrix} \hat{a_2} \\ \hat{U} + K\hat{V} \\ \hat{a_N} \end{vmatrix}}_{\substack{(N-1)\times(N-1) \text{ MATRIX} \\ \text{OF SAME TYPE}}} \leftarrow \widehat{U+KV}$$

INDUCTION
$$= \sum_{j=1}^{N} (-1)^{1+j} A_{1j} \left(\begin{vmatrix} \hat{a_2} \\ \hat{U} \\ \hat{a_N} \end{vmatrix} + K \begin{vmatrix} \hat{a_2} \\ \hat{V} \\ \hat{a_N} \end{vmatrix} \right)$$

$$= \left(\sum_{j=1}^{N} (-1)^{1+j} A_{1j} \begin{vmatrix} \hat{a_2} \\ \hat{U} \\ \hat{a_N} \end{vmatrix} \right) + K \left(\sum_{j=1}^{N} (-1)^{1+j} A_{1j} \begin{vmatrix} \hat{a_2} \\ \hat{V} \\ \hat{a_N} \end{vmatrix} \right)$$

DEF
$$= \begin{vmatrix} a_1 \\ \vdots \\ U \\ \vdots \\ a_N \end{vmatrix} + K \begin{vmatrix} a_1 \\ \vdots \\ V \\ \vdots \\ a_N \end{vmatrix} \quad \blacksquare$$

II – COFACTOR EXPANSION

(NOW LET ME SHOW YOU WHY YOU CAN EVALUATE THE DET BY CHOOSING ANY ROW)

THEOREM For all $i = 1, \ldots, N$,

$$|A| = \sum_{j=1}^{N} (-1)^{i+j} A_{ij} \, |\tilde{A}_{ij}|$$

(COFACTOR EXPANSION ALONG ROW i)

WHY? $|A| = \begin{vmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_N \end{vmatrix}$

BUT $a_i = (A_{i1}, A_{i2}, \ldots, A_{iN}) = A_{i1} e_1 + \cdots + A_{iN} e_N$

SO $|A| = \begin{vmatrix} a_1 \\ \vdots \\ A_{i1} e_1 + \cdots + A_{iN} e_N \\ \vdots \\ a_N \end{vmatrix}$ $\leftarrow i$

MULTILINEARITY

$$= A_{i1} \begin{vmatrix} a_1 \\ \vdots \\ e_1 \\ \vdots \\ a_N \end{vmatrix} + \cdots + A_{iN} \begin{vmatrix} a_1 \\ \vdots \\ e_N \\ \vdots \\ a_N \end{vmatrix}$$

$$= \sum_{j=1}^{N} A_{ij} \begin{vmatrix} a_1 \\ \vdots \\ e_j \\ \vdots \\ a_N \end{vmatrix} \leftarrow i^{\text{TH}} \text{ ROW}$$

$$\begin{array}{cc} & j \\ i & \left[\begin{array}{c|c} \tilde{A}_{ij} & \\ \hline 0\,0\,0 & 0\,0\,0 \end{array}\right] \end{array}$$

KEY LEMMA (LAST TIME)

$$= \sum_{j=1}^{N} A_{ij} (-1)^{i+j} \tilde{A}_{ij} \checkmark$$

(SO FROM NOW, FEEL FREE TO EXPAND OUT DET ALONG ANY ROW YOU WISH; BUT NOT ANY COLUMN YET!)

III — DET & ROW-REDUCTION

(FINALLY, LET'S SEE THE EFFECTS OF ROW-REDUCTION ON THE DET; WILL BE CRUCIAL NEXT TIME WHEN WE'LL SHOW $|AB| = |A||B|$)

LEMMA IF A HAS 2 IDENTICAL ROWS, THEN $|A| = 0$

EX

$$\begin{vmatrix} 2 & 3 & 2 \\ 4 & 1 & 4 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

(PROOF IS BY STRAIGHTFORWARD INDUCTION, WHERE YOU EXPAND ALONG A NONIDENTICAL ROW, IF ANY)

THEOREM [TYPE I] FOR ANY r, s

$$\begin{vmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_N \end{vmatrix} = - \begin{vmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_N \end{vmatrix}$$

WHY? CONSIDER

2 IDENTICAL ROWS, MULTILINEAR

$$0 = \begin{array}{c} \\ r \\ s \\ \\ \end{array} \begin{vmatrix} a_1 \\ a_r + a_s \\ a_r + a_s \\ a_N \end{vmatrix} = \begin{array}{c} \\ r \\ s \\ \\ \end{array} \begin{vmatrix} a_1 \\ a_r \\ a_r + a_s \\ a_N \end{vmatrix} + \begin{array}{c} \\ r \\ s \\ \\ \end{array} \begin{vmatrix} a_1 \\ a_s \\ a_r + a_s \\ a_N \end{vmatrix}$$

MULTILINEAR

$$= \underbrace{\begin{vmatrix} a_1 \\ a_r \\ a_r \\ a_N \end{vmatrix}}_{0} + \begin{vmatrix} a_1 \\ a_r \\ a_s \\ a_N \end{vmatrix} + \begin{vmatrix} a_1 \\ a_s \\ a_r \\ a_N \end{vmatrix} + \underbrace{\begin{vmatrix} a_1 \\ a_s \\ a_s \\ a_N \end{vmatrix}}_{0} = 0$$

$$\Rightarrow \begin{vmatrix} a_1 \\ a_r \\ a_s \\ a_N \end{vmatrix} = - \begin{vmatrix} a_1 \\ a_s \\ a_r \\ a_N \end{vmatrix}$$

THEOREM [TYPE 2]

$$\begin{vmatrix} a_1 \\ K a_r \\ a_N \end{vmatrix} = K \begin{vmatrix} a_1 \\ a_r \\ a_N \end{vmatrix}$$

(THIS IS JUST MULTILINEARITY)

THEOREM [TYPE 3]

$$(\times K) \begin{vmatrix} a_1 \\ a_r \\ a_s \\ a_N \end{vmatrix} = \begin{vmatrix} a_1 \\ a_r \\ a_s + K a_r \\ a_N \end{vmatrix}$$

WHY?

$$\begin{vmatrix} a_1 \\ a_r \\ a_s + K a_r \\ a_N \end{vmatrix} \overset{\text{MULTI}}{=} \begin{vmatrix} a_1 \\ a_r \\ a_s \\ a_N \end{vmatrix} + K \underbrace{\cancel{\begin{vmatrix} a_1 \\ a_r \\ a_r \\ a_N \end{vmatrix}}}_{0} = \begin{vmatrix} a_1 \\ a_r \\ a_s \\ a_N \end{vmatrix} \checkmark$$

NOTE CAN USE THAT TO CALCULATE DET:

EX

$$\begin{matrix} (\times -4) \\ \\ (\times -7) \end{matrix} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = (-3)(-6) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

IV – DETERMINANTS AND INVERTIBILITY (IF TIME PERMITS)

FACT: IF A $(N \times N)$ IS NOT INVERTIBLE, THEN $|A| = 0$

COROLLARY: $|A| \neq 0 \Rightarrow A$ INV

(WILL SEE CONVERSE NEXT TIME)

WHY? A NOT INV $\Rightarrow$ RANK$(A) < N$

$\Rightarrow$ A HAS $< N$ PIVOTS

$\Rightarrow$ ROWS OF A ARE LIN DEP

(OTHERWISE PIVOT IN EACH ROW)

$\Rightarrow$ ONE ROW (WLOG a_1) IS A LINEAR COMBO OF THE OTHER ROWS

$a_1 = c_2 a_2 + \cdots + c_N a_N$

THEN
$$|A| = \begin{vmatrix} a_1 \\ \vdots \\ a_N \end{vmatrix}$$

$$= \begin{vmatrix} c_2 a_2 + \cdots + c_N a_N \\ a_2 \\ \vdots \\ a_N \end{vmatrix}$$

MULTILINEAR
$$= c_2 \begin{vmatrix} a_2 \\ a_2 \\ \vdots \\ a_N \end{vmatrix} + \cdots + c_N \begin{vmatrix} a_N \\ a_2 \\ \vdots \\ a_N \end{vmatrix}$$

0   0

$= 0$ ■