WEDNESDAY, MAY 29, 2019

LECTURE 25 – PROPERTIES OF DETERMINANTS (SECTION 4.3)

:) WELCOME TO OUR THIRD AND FINAL INSTALLMENT OF OUR DETERMINANT TRILOGY! TODAY WE'LL PROVE ALL THE CUTE PROPERTIES OF DETERMINANTS YOU USED IN MATH 3A, STARTING WITH:

I – $\det(AB) = \det(A)\det(B)$

THEOREM $\det(AB) = \det(A)\det(B)$ (IF A, B ARE $n \times n$)

(VERY NEAT PROOF B/C IT DOESN'T USE ANY MESSY COMPUTATION AT ALL, JUST SOME NEAT LINEAR ALGEBRA)

NOTE IF A IS NOT INVERTIBLE, THEN $|A| = 0$ (LAST TIME)

BUT THEN AB IS NOT INV (SECTION 2.4)

SO $|AB| = 0 = |A||B|$ ✓

SO ASSUME A IS INVERTIBLE

CASE 1 A IS AN ELEMENTARY MATRIX, SAY OF TYPE 1

(= INTERCHANGE 2 ROWS, $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$)

BUT THEN AB INTERCHANGES 2 ROWS OF B, SO $|AB| = -|B|$ (LAST TIME)

BUT $|A| = -1$ (SEE HW)

SO $|AB| = -|B| = |A||B|$ ✓

(SIMILAR FOR TYPES 2 & 3)

CASE 2 GENERAL CASE

IN THAT CASE, SINCE A IS INVERTIBLE, A IS A PRODUCT OF ELEM. MATRICES,

$$A = E_m \cdots E_2 E_1 \qquad E_i \text{ ELEM}$$

BUT THEN $$|AB| = |\overset{\text{ELEM}}{E_m} E_{m-1} \cdots E_1 B|$$

CASE I $$= |E_m| \, |\overset{\text{ELEM}}{E_{m-1}} \cdots E_1 B|$$

CASE I $$= |E_m| \, |E_{m-1}| \cdots |\overset{\text{ELEM}}{E_2}| \, |E_1| \, |B|$$

CASE I (IN REVERSE) $$= |E_m| \, |E_{m-1}| \cdots |\overset{\text{ELEM}}{E_3}| \, |E_2 E_1| \, |B|$$

CASE I $$= |E_m E_{m-1} \cdots E_2 E_1| \, |B|$$

$$= |A| \, |B|$$

II - DET(A^{-1})

COROLLARY A INVERTIBLE $\iff |A| \neq 0$

IN THAT CASE $|A^{-1}| = \dfrac{1}{|A|}$

WHY? $(\Leftarrow)$ IF A IS NOT INVERTIBLE, THEN $|A| = 0$ (WRITTE

$(\Rightarrow)$ IF A IS INV, THEN $A^{-1} A = I$

SO $$|A^{-1} A| = |I| = 1 \Rightarrow \overbrace{|A^{-1}|}^{\neq 0} \, \overbrace{|A|}^{\neq 0} = 1$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|}$$

III - DET(A^T)

THEOREM $|A^T| = |A|$

WHY? NOTE IF A IS NOT INV, THEN RANK$(A) < N$

SO RANK(A^T) = RANK$(A) < N$

SO A^T IS NOT INV

SO $|A^T| = 0 = |A|$ ✓

ASSUME A IS INV, THEN A IS A PRODUCT OF ELEM MATRICES

$$A = E_m \cdots E_2 E_1$$

THEN $A^T = (E_m \cdots E_2 E_1)^T = E_1^T E_2^T \cdots E_m^T$

SO $|A^T| = |E_1^T \cdots E_m^T|$

$= |E_1^T| \cdots |E_m^T|$; $|E_i^T| = E_i$ FOR ALL i (SEE HW)

$= |E_1| \cdots |E_m|$

$= |E_m| \cdots |E_1|$

$= |E_m \cdots E_1|$

$= |A|$

COROLLARY CAN EVALUATE $|A|$ ALONG ANY COLUMN OF A

WHY? THE j^{TH} COLUMN OF A IS THE j^{TH} ROW OF A^T

SO (EXPANSION ALONG COL j OF A) = (" ROW j " A^T)

$= |A^T|$

$= |A|$ ✓

IV – CRAMER'S RULE

(THE REASON DETERMINANTS ARE SO AWESOME IS NOT JUST B/C OF THEIR PROPERTIES, BUT ALSO B/C OF THEIR APPLICATIONS. I REALLY COULD SPEND DAYS TALKING ABOUT THE APPLICATIONS, BUT LET ME JUST FOCUS ON ONE OF THEM. NAMELY, YOU CAN USE DETERMINANTS TO SOLVE SYSTEMS OF EQUATIONS IN A SNAP)

EX SOLVE $A\underline{x} = \underline{b}$, $A = \begin{bmatrix} -5 & 3 \\ 3 & -1 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$

M_1 = A BUT WITH $\underline{b}$ IN FIRST COL

$$x_1 = \frac{\begin{vmatrix} 9 & 3 \\ -5 & -1 \end{vmatrix}}{\underbrace{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}}_{|A|}} = \frac{6}{-4} = -\frac{3}{2} \qquad \left(x_1 = \frac{|M_1|}{|A|} \right)$$

$$x_2 = \frac{\overbrace{\begin{vmatrix} -5 & 9 \\ 3 & -5 \end{vmatrix}}^{|M_2|}}{\underbrace{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}}_{|A|}} = \frac{-2}{-4} = \frac{1}{2} \qquad \left(x_2 = \frac{|M_2|}{|A|} \right)$$

ANS $\underline{x} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$ (WOOOOW! AND PEOPLE SAY THEY HATE DET...)

(1 – SECOND WAY OF SOLVING A SYSTEM! THAT SAID, IN PRACTICE IT'S A PAIN B/C FOR A 3x3 SYSTEM, YOU HAVE TO SOLVE FOR 4 3x3 DETERMINANTS)

WHY THIS WORKS

FACT IF $A\underline{x} = \underline{b}$, AND M_k = A BUT $\underline{b}$ IN k^{TH} COL, THEN $x_k = \dfrac{|M_k|}{|A|}$

WHY?

STEP 1 LET $\underline{X}_K = \underline{I}$ BUT WITH x IN K^{TH} COL

$= [e_1 \cdots x \cdots e_N]$

EX $K = 2$, $\underline{X}_2 = \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}$

NOTE EXPANDING ALONG ROW K, WE GET $\underline{X}_K = \begin{bmatrix} 1 & & x_1 & & \\ & 1 & \vdots & & \\ 0 \; 0 & & x_K & 0 & 0 \\ & & \vdots & 1 & \\ & & x_N & & 1 \end{bmatrix}$

$|\underline{X}_K| = (-1)^{K+K} x_K \begin{vmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{vmatrix} = x_K$

$x_K = |\underline{X}_K|$

STEP 2 BUT $A\underline{X}_K = A[e_1 \cdots x \cdots e_N]$

$= [Ae_1 \cdots Ax \cdots Ae_N]$, $A = [a_1 \cdots a_N]$

$= [a_1 \cdots b \cdots a_N]$

$= A$ BUT b IN K^{TH} COL

$= M_K$

SO $A\underline{X}_K = M_K$

SO $|A\underline{X}_K| = |M_K| \Rightarrow |A|\,|\underline{X}_K| = |M_K| \Rightarrow |\underline{X}_K| = \dfrac{|M_K|}{|A|}$

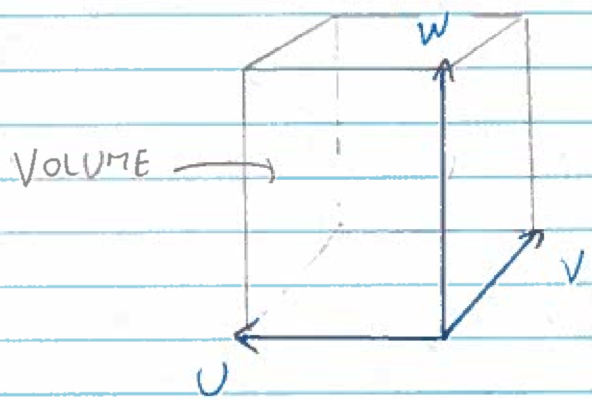
$x_K = |\underline{X}_K| = \dfrac{|M_K|}{|A|}$ ∎

STEP 1

(V – PARALLELIPIPEDS)

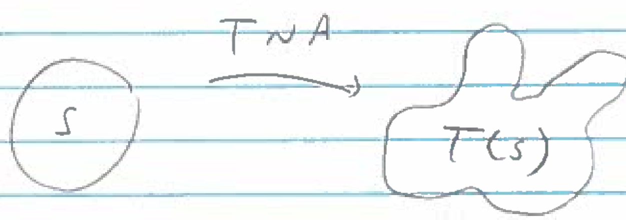
NOTE 1) VOLUME OF PARALLELIPIPED DETERMINED BY U, V, W

$$= \left| \det \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right|$$



2) $$\frac{\det \begin{bmatrix} U \\ V \\ W \end{bmatrix}}{\left| \det \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right|} = O \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \text{ORIENTATION OF PARALLELIPIPED}$$

3) IN GENERAL, IF T IS A LINEAR TRANSFORMATION AND S IS A SET



THEN $\boxed{\text{Vol}(T(S)) = |\det(A)|\ \text{Vol}(S)}$