FRIDAY, MAY 31, 2019

LECTURE 26 – EIGENVALUES AND EIGENVECTORS (I) (SECTION 5.1)

) HELLO EVERYONE AND WELCOME TO THE FINAL CHAPTER OF THE COURSE! THIS CHAPTER IS ALL ABOUT EIGENVALUES / EIGENVECTORS, WHICH ARE SPECIAL VALUES THAT YOU CAN ATTACH TO A MATRIX / LT.

I – DEFINITION

DEF IF A IS $N \times N$ AND $\boxed{AV = \lambda V}$ FOR SOME $V \neq 0$, THEN:

1) λ IS CALLED AN EIGENVALUE OF A
2) V " EIGENVECTOR OF A CORRESP. TO λ

EX $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$AV = \underbrace{\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{V} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} = \underbrace{5}_{\lambda}\underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{V} = \lambda V$$

$\lambda = 5$ IS AN EIGENVALUE, $V = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ AN EIGENVECTOR CORRESP TO $\lambda = 5$

(THIS NOTION CAN BE EASILY GENERALIZED TO ABSTRACT LT, AND IN FACT IN THIS CHAPTER WE'LL FREQUENTLY USE BOTH FORMULATIONS)

DEF IF $T: V \to V$ LT AND $T(V) = \lambda V$ FOR SOME $V \neq 0$ THEN

1) λ EIGENVALUE OF T
2) V EIGENVECTOR OF T CORRESP TO λ

EX $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ (SMOOTH FUNCTIONS)

$T(y) = y'$

IF $y = e^{2x}$, THEN $\underbrace{T(y)}_{T(v)} = (e^{2x})' = 2e^{2x} = \underbrace{2y}_{\lambda v}$

SO $\lambda = 2$ IS AN EIGENVALUE AND $v = e^{2x}$ IS AN EIGENVECTOR (EIGENFUNCTION)

NOTE HOW ARE THE TWO DEF RELATED?

FACT v IS AN EIGENVECTOR OF $A \iff v$ IS AN EIGENVECTOR OF L_A

WHY? ($\Rightarrow$) LET $T = L_A$, THEN $T(v) = L_A(v) = Av = \lambda v$, SO $T(v) = \lambda v$

($\Leftarrow$) SIMILAR

II - FINDING EIGENVALUES

(NOW THAT WE KNOW WHAT AN EIGENVALUE / EIGENVECTOR ARE, OUR NEXT JOB IS TO FIGURE OUT HOW TO FIND THEM, LET'S START WITH EIGENVALUES)

THEOREM λ IS AN EIGENVALUE OF $A \iff \boxed{\det(A - \lambda I) = 0}$

WHY? $Av = \lambda v$ FOR $v \neq 0$

$\iff Av - \lambda v = 0$

$\iff Av - \lambda I v = 0$

$\iff (A - \lambda I)v = 0$ FOR $v \neq 0$

$\iff N(A - \lambda I) \neq \{0\}$

$\iff A - \lambda I$ NOT INV

$\iff \det(A - \lambda I) = 0$

DEF $\boxed{f(t) = \det(A - tI)}$ IS CALLED THE CHARACTERISTIC POLY OF A

EX $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$f(t) = |A - tI| = \begin{vmatrix} 1-t & 1 \\ 4 & 1-t \end{vmatrix} = (1-t)^2 - 4 = t^2 - 2t + 1 - 4$

$= t^2 - 2t - 3$

$f(t) = t^2 - 2t - 3 = (t-3)(t+1) = 0 \Rightarrow t = 3, -1$

EIGENVALUES $\lambda = -1, 3$

SOME FACTS

1) $f(t)$ IS A POLY OF DEGREE N (A IS $N \times N$)

2) ZEROS OF f ARE THE EIGENVALUES OF A

3) A CANNOT HAVE MORE THAN N EIGENVALUES

(OTHERWISE f WOULD BE A DEG N POLY WITH $N+1$ ZEROS)

III - EIGENVALUES OF T

(WHAT ABOUT A MORE ABSTRACT LT T? MAN, IF ONLY WE COULD ASSOCIATE A MATRIX TO A LT! BUT WE CAN !!!)

DEF IF $T : V \to V$ AND $A = [T]_\beta^\beta$ (β = ANY BASIS OF V)

THEN THE CHAR POLY OF T IS :

$$\boxed{f(t) = \det(A - tI)}$$

⚠ INDEPENDENT OF THE CHOICE OF β ! (SEE BELOW)

EX $V = P_2(\mathbb{R})$, $T(p) = p + (x+1)p'$

LET $\beta = \{1, x, x^2\}$ STANDARD BASIS OF V, FIND $A = [T]_\beta^\beta$

$T(1) = 1 + (x+1)(0) = 1 = (1)(1) + 0x + 0x^2$

$T(x) = x + (x+1)(1) = 2x+1 = (1)(1) + 2x + 0x^2$

$T(x^2) = x^2 + (x+1)(2x) = 3x^2+2x = (0)(1) + 2x + 3x^2$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$f(t) = \det(A - tI) = \begin{vmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{vmatrix} = (1-t)(2-t)(3-t) = 0$$

EIGENVALUES $\lambda = 1, 2, 3$

WHY INDEPENDENT OF CHOICE OF BASIS?

LET γ BE ANOTHER BASIS OF V AND $B = [T]_\gamma^\gamma$

RECALL (SECTION 2.5) $\quad [T]_\gamma^\gamma = Q\,[T]_\beta^\beta\,Q^{-1} \qquad Q = Q_{\gamma \leftarrow \beta}$

$B = Q A Q^{-1}$

THEN $\det([T]_\gamma^\gamma - tI) = \det(B - tI)$

$= \det(QAQ^{-1} - tI)$

$= \det(QAQ^{-1} - tQQ^{-1})$

$= \det(Q(A-tI)Q^{-1})$

$= \cancel{|Q|}\,|A-tI|\,\cancel{|Q^{-1}|}$

$= \det(A-tI) = \det([T]_\beta^\beta - tI)$

so $\det\left([T]_\gamma - tI\right) = \det\left([T]_\beta - tI\right) = f(t)$

(IN PARTICULAR THE EIGENVALUES STAY THE SAME)

IV – FINDING EIGENVECTORS

FACT v IS AN EIGENVECTOR OF T CORRESP TO λ

IFF $\boxed{v \in N(T - \lambda I)}$

WHY? $Tv = \lambda v \iff Tv - \lambda v = 0$

$\iff Tv - \lambda I v = 0$

$\iff (T - \lambda I)v = 0$

$\iff v \in N(T - \lambda I)$

SIMILARLY v " OF A " $\lambda \iff v \in \mathrm{Nul}(A - \lambda I)$

EX FIND THE EIGENVECTORS OF $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

FOUND $\lambda = 3, -1$, $A - tI = \begin{bmatrix} 1-t & 1 \\ 4 & 1-t \end{bmatrix}$

$\lambda = 3$ $\mathrm{Nul}(A - 3I) = \mathrm{Nul}\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} = \cdots = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$\lambda = -1$ $\mathrm{Nul}(A - (-1)I) = \mathrm{Nul}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \cdots = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$

NOTE CAN CHECK THAT $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ IS A BASIS OF $\mathbb{R}^2$

SO GET A BASIS OF $V = \mathbb{R}^2$ CONSISTING OF EIGENVECTORS OF A