MATH 121A – MIDTERM STUDY GUIDE

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GENERAL INFO

The Midterm Exam takes place on **Friday, May 3**, during the usual time (11-11:50 am) and the usual place (PSCB140). It's possible that I'll give you 5 minutes extra, so make sure you can stay until 11:55 am. Please bring your student ID, as we'll be checking IDs during the exam. There will be a seating chart for the exam, which I'll send out a day or two before the exam.

This is the study guide for the exam. Please read it carefully, because it contains a lot of info about what's going to be on the exam and what I expect you to know or not to know. That said, remember that this study guide is just a *guide* and not a complete list. I've tried to make this list as complete as possible, but there are always things that I may have missed.

The midterm covers sections 1.2-1.6 and 2.1-2.4 inclusive.

There will be 4 problems on the exam. The problems will be a mix of definitions, computational problems, homework problems (*including* the optional problems), proofs of theorems you need to know, and problems not on the homework. If you want to have an idea of the format of the exam, please look at the practice exam. But don't be fooled, the actual exam might have different questions.

IMPORTANT: On the exam, you are graded not only on the correct answer, but also on the way you write out your answer. Don't be surprised if you get lots of points off if your proof is too short or if you skip too many details! I will be especially picky about this, even more so than the homework grader or your TA or other instructors you may have had. Please look at the solutions of the practice exam to get an idea of how I want you to write out your answers. Here less is **NOT** more; if you have any doubts of whether you need to justify a part of the answer, do it! And write in complete sentences, not just mathematical symbols. It might seem weird now, but being able to communicate your answer clearly will be extremely useful

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in later math classes.

Beware: In lecture, I tried to hit the main points of each section, but there are topics in the book and/or the homework that I didn't cover in lecture but that I expect you to know for the exam. That's why it's especially important that you look at this study guide, in order to avoid surprises. The *best* way to prepare for the exam is to read the book and go over the homework problems. You don't need to study the quizzes or the discussion worksheets, since I didn't look at them at all.

Finally, remember that I'm on your side: This class will be curved, in the sense that I will take all your raw scores, add them up, and then curve that total raw score. So the harder the midterm, the more generous the curve. I won't curve individual exams. You can find the official grading percentages on the syllabus.

12 PROOFS YOU ABSOLUTELY NEED TO KNOW

Know the statements **and** the proofs of the following 12 theorems. You are *guaranteed* to have one of those 12 proofs on the exam.

Warning: Although I won't explicitly ask you to reprove theorems not on this list, you'll still need to know their statements and how to use them, so it's still worthwile to look at them because they use important techniques covered in this course.

Theorem	Name	YouTube Video
1.4 in 1.3	Intersection of subspaces	
1.5 in 1.4	Span is a subspace	Video
1.7 in 1.5	Intruder Theorem	Video
(1.10 in 1.6)	(Replacement Theorem: Statement but not proof)	Video
Cor 1 and 2(b)(c) in 1.6	Corollaries of Replacement Theorem	Video
2.1 in 2.1	$N(T)$ and $R(T)$ are subspaces	Video
2.3 in 2.1	Dimension Theorem/Rank-Nullity Theorem	Video
2.5 in 2.1	One-to-one is equivalent to onto	
2.6 in 2.1	Linear Extension Theorem	Video
2.11 in 2.3	Matrix of UT, the proof is on page 87	Video
2.18 in 2.4	Matrix of T^{-1}	Video
2.19 in 2.4	Isomorphic iff same dimension	Video
2.20 in 2.4	Φ is an isomorphism	Video

YouTube Playlists

There are a bunch of videos on my YouTube channel, based on the concepts covered in this course. Check them out if you need help with a topic:

- Chapter 1: Vector Spaces
- Chapter 2: Linear Transformations and Matrices

Note: 1.1.11 means Problem 11 in section 1.1.

Section 1.2: Vector Spaces

- You do **NOT** need to know the axioms of vector spaces (VS1)-(VS8), just know how to use them
- Look at Theorem 1.1 (Cancellation Law for Vector Addition), Corollary 1, Corollary 2, and Theorem 1.2(a) and (c). You don't need to memorize their proofs.
- Ignore the Additional Problem on HW 1
- Know how to show something is not a vector space (1.2.17-18)
- If you need more examples of vector spaces, check out this video: 25 Ex of VS

Section 1.3: Subspaces

- Know the definition of a subspace and the statement of Theorem 1.3 (but not its proof)
- Know how to show that something is or is not a subspace (for example, see 1.3.10) Also check out the examples in Lecture 2, as well as those videos: Subspace and Not Subspace.
- Know the statement and proof of Theorem 1.4 about intersection of subspaces
- Give an example showing the union of subspaces is not necessarily a subspace, also look at 1.3.19 and this video: Union of subspaces
- Know how to carry out a proof by induction, see 1.3.20

Section 1.4: Linear Combinations and Systems of Linear Equations

- Know the definition of linear combination
- Ignore Example 1 in the book
- **Note:** On the exam, you're allowed to use the row-reduction technique you learned in 3A, but *only* to solve a system of equations. You're not allowed to use any facts about pivots, and you're not allowed to use row-reduction to find the inverse of a matrix. For a review, check out Gaussian elimination

- Determine if a vector is a linear combination of other ones (Example 2 in the book, and 1.4.5)
- Know the definition of the span of a set and know how to prove Theorem 1.5, also see Span is a subspace
- Know the definition of spanning set of V
- Figure out from the definition whether vectors span a given space (Examples 3-5, and 1.5.10). I won't ask anything too crazy about it because we'll find a better technique in chapter 3. A good example is the last example I did in Lecture 3 (on pages 5-6)
- Also check out 1.5.13 and 1.5.15

Section 1.5: Linear Dependence and Linear Independence

- Know the definition of linear dependence and linear independence
- Know how to show whether a set if linearly dependent or linearly independent. Again, feel free to use row-reduction for it. Check out Linear Independence if you want.
- Know the statement and Theorem 1.6 and its Corollary, but you don't need to memorize its proof
- Know the statement and proof of the "Intruder Theorem" (Theorem 1.7). Intuitively it's saying that if you add a vector $\mathbf{v}$ to a linearly independent set, and if that set becomes linearly dependent, then it's $\mathbf{v}$'s fault. See this video for the proof: Intruder Theorem
- Also check out 1.5.10, 1.5.14 and 1.5.15 (if you want), but ignore 1.5.20

Section 1.6: Bases and Dimension

- Know the definition of a basis and how to show something is a basis (like in 1.6.3)
- Know the statement and proof of Theorem 1.8
- Give examples of bases of vector spaces
- Find a basis of the span of a set, as in Example 6 or in 1.6.7 or in Problem 1 of the practice exam. I won't give a very complicated problem because we'll find an easier way of doing this in Chapter 3.
- Know the statement, but not the proof of Theorem 1.9. You might also want to check out the proof I gave in Lecture 5, just as an example of how to use induction.
- Know the statement of the Replacement Theorem (Theorem 1.10). You do **NOT** need to know the proof of the replacement theorem. See this video for an explanation of the replacement theorem: Replacement Theorem.

- Know how to use the Replacement Theorem, in particular know the statements and proof of Corollary 1 and Corollary 2(b)(c) in the book. Also know how to do AP1 in Homework 3. You don't need to know the proof of Corollary 2(a), but know its statement. See the above video for an explanation.
- **Note:** Corollary 2(c) and the Corollary on page 51 are very important and will be used throughout the course
- Know the definition of dimension and explain why the definition makes sense. See this video for an overview: Dimension.
- Use the notion of dimension to your advantage, as in 1.6.4
- Find the dimension of a subspace, like in 1.6.14 or 1.6.16
- Give an example of an infinite-dimensional vector space, as in Infinite Dimensions
- Unless otherwise specified, do **NOT** assume your vector space is finite-dimensional
- Know the statement of Theorem 1.11, bur not its proof.
- **IGNORE** the section on the Lagrange Interpolation Formula, but if you're curious, check out: Lagrange Interpolation Formula
- Don't forget about 1.6.29. You do **NOT** need to know the definition of sum or direct sums of subspaces; I would give you that definition if necessary. A good practice problem would be: Direct Sums
- Section 1.7 (Maximal Linearly Independent Subsets) will **NOT** be on the exam, but if you're curious how to show any vector space has a basis, check out: Maximal Linearly Independent Subsets

SECTION 2.1: LINEAR TRANSFORMATIONS, NULL SPACES, AND RANGES

- Know the definition of a linear transformation, and know the statement (but not the proof) of properties $1 - 4$ on page 65
- Know how to show that T is linear (Example 1 or Example 6) or not linear (2.1.9). Check out Linear Transformations
- Ignore Example 2
- Know the definition of $N(T)$ and $R(T)$, the statement and proof of Theorem 2.1, the statement but not the proof of Theorem 2.2, and know how to find bases for $N(T)$ and $R(T)$ (Example 10). See also Nullspace is a Subspace.
- Know the definition of the rank and the nullity of T and know how to calculate them
- Know the statement and the proof of the Dimension Theorem/Rank-Nullity Theorem (Theorem 2.3, or see Rank-Nullity Theorem Proof), and know the statement and proof of Theorem 2.5, which is a cool

consequence of the Rank-Nullity Theorem, and check out Examples 11 and 12

- Know the definition of one-to-one and know the statement but not the proof of Theorem 2.4
- Know the statement and proof of Theorem 2.6, also known as the Linear Extension Theorem (I did this at the end of Lecture 8, see also Linear Extension Theorem). It's used everywhere in Linear Algebra, especially look at Problem 4 of the practice exam. Also remember the Corollary on page 73.
- 2.1.2, 2.1.6, and 2.1.15-16, 2.1.21, and AP2 on Homework 4 are good review problems, as well as this video One-to-one and Onto and I highly recommend looking at 2.1.13 and 2.1.14(c)
- You don't need to know the definition of T-invariant (as in 2.1.28)
- Ignore problems 2.1.37 and 2.1.39, as well as AP2 on Homework 3 and AP1 on Homework 4
- Also check out the following videos (based on homework problems): Intersection of Range, Linear Transformation with a given range, and Derivative and Linear Independence

SECTION 2.2: THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

- Know the definition of the coordinate vector of $\mathbf{x}$ relative to a basis β (page 80)
- Find the coordinates of $\mathbf{x}$ with respect to β. Again, totally ok to use row-reduction here. Also see Coordinates
- Know the definition of the matrix of T, $A = [T]_\beta^\gamma$ (bottom of page 80). Know both formulations, the one with $T(\mathbf{v_j})$ as a sum (right above the definition) and the one in terms of j-th column of A is $[T(\mathbf{v_j})]_\gamma$ (right below the definition)
- Know how to find the matrix of T relative to a basis (see 2.2.5, 2.2.10, Derivative in a Box, Matrix of a Matrix, Matrix with respect to a basis)
- Know the definition of sum and scalar multiplication of linear transformations (page 82) and know the statement of Theorem 2.7, but not its proof
- Know the definition of $\mathcal{L}(V, W)$ and $\mathcal{L}(V)$
- Know the statement, but not the proof of Theorem 2.8, as well as Example 5
- Also check out 2.2.13, 2.2.14, 2.2.16

SECTION 2.3: COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION

- Know the definition of the composition UT of U and T and know the statement and the proof of Theorem 2.11 (the proof is on page 87), and the statements but not the proofs of Theorems 2.9 and 2.10 and the Corollary on page 89.
- Know the definition of AB and know how to derive it (page 87, also see Where Matrix Multiplication comes from, although I'm using slightly different notation in the video)
- Know how to calculate AB, see Matrix Multiplication, and remember that in general, $AB \neq BA$, see AB vs. BA
- Know the definition of A^T and know how to show $(AB)^T = B^T A^T$
- Know the properties in Theorem 2.12, but don't worry about the proof.
- For Theorem 2.13, you don't need to know the statement or the proof, but know how to use it. I clarified the statements a bit at the very end of Lecture 10.
- Know the statement but not the proof Theorem 2.14. You might want to look at the proof I gave at the beginning of Lecture 11. Also look at Example 3 to see how to use it.
- Know the definiton of L_A and the statements but not the proofs of Theorem 2.15
- Also know the statement but not the proof of Theorem 2.16; look at how cool it is!
- Ignore the section on Applications
- Also look at 2.3.9, 2.3.11, 2.3.12(a)(b), and 2.3.13
- Optional, but really cool: $A^2 = A$, $A^2 = O$, $A^2 = I$

SECTION 2.4: INVERTIBILITY AND ISOMORPHISMS

- Know the definitions of 'inverse of T' and 'T invertible'
- Know how to show that T is invertible (see 2.4.2)
- Know properties 1 and 2 on page 100, but you don't need how to prove them, and ignore property 3
- Know the statement but not the proof of Theorem 2.17. You can use either the book's proof of the proof I gave in Lecture 12
- Know the definition of 'A invertible' and A^{-1}
- Show that A^{-1} is unique (see statement at the bottom of page 100)
- Know the statement but not the proof of the Lemma on page 101
- Remember that if $\dim(V) = \dim(W) < \infty$, then one-to-one is equivalent to onto, that should be useful

- Know the statement and proof of Theorem 2.18, as well as the statements (but not the proofs) of Corollaries 1 and 2; see this video: Matrix of T-1
- Know the definition of isomorphism
- Know how to show that two vector spaces are isomorphic, both by directly writing down an isomorphism and showing that it's linear, one-to-one, and onto (2.4.14 and Isomorphism), or by counting the dimensions of V and W (see Theorem 2.19, or 2.4.3)
- Know the statement and proof of Theorem 2.19, see Isomorphism and Dimension
- Know the statement and proof of Theorem 2.20
- You don't need to know the definition of the standard representation (page 104), and you don't need to know the formula in the middle of page 105, but you need to know the statement and proof of Theorem 2.21 and to understand the diagram in Figure 2.2 and Example 7, see the following videos: 2 Miracles of Linear Algebra and Differentiate with Linear Algebra
- Know how to calculate T^{-1}, see this video for an example: Calculate T^{-1}
- Check out 2.4.4, 2.4.9, 2.4.10(a)(b)