LECTURE 28 - DIAGONALIZABILITY (I) (SECTION 5.4)

Previously on Captain Peyamnica, we learned about diagonalization, which is a near way of transforming a matrix into a diagonal matrix.

Recall $A$ is diagonalizable $\iff$ there is a basis of $\mathbb{F}^n$ consisting of eigenvectors of $A$.

Main: How to concretely check if $A$ is diagonalizable.

(Note: all the results are true with for LT $T$ and matrices $A$.

I - Key Lemma will frequently switch between the two)

It all relies on the following key lemma:

Theorem: If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of $T$ and $v_1, \ldots, v_n$ are the corresponding eigenvectors, then $\{v_1, \ldots, v_n\}$ is LI

(“Eigenvectors corresponding to $\neq$ eigenvalues are LI”)

Ex: $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\begin{array}{l} \lambda_1 = -1 \implies \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \ \ \ \ \lambda_2 = 3 \implies \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$

This says $\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ is automatically LI (why? a basis of $\mathbb{R}^2$)

Why? induction on $n$

1) Base ($n=1$) $\{v_1\}$ is LI. (since $v \neq 0$)

2) Ind. Suppose $Pn-1$ is true, show $Pn$ is true.

Let $v_1, \ldots, v_n$ be eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$ (distinct)

And suppose $a_1 v_1 + \cdots + a_n v_n = 0$ (**)
3) On the one hand, apply $T$ to $+$:

$$T(a_1 v_1 + \cdots + a_n v_n) = T(0) = 0$$

$$a_1 T(v_1) + \cdots + a_n T(v_n) = 0$$

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad (1)$$

4) On the other hand, multiply $+$ by $1$, (THICK!)

$$1 \cdot (a_1 v_1 + \cdots + a_n v_n) = 1 \cdot 0$$

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad (2)$$

5) Subtract the two:

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

$$-(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = 0$$

$$a_2 (v_1 - v_2) + \cdots + a_n (v_n - v_1) = 0$$

6) By induction hyp., $(v_1, \ldots, v_n)$ are $(\perp, \perp)$ so

Since $a_1, \ldots, a_n$ are distinct

$$a_1 (v_1 - v_1) = 0$$

$$\Rightarrow a_1 = 0$$

Therefore, $a_1 v_1 = 0 \Rightarrow a_1 = 0$ (since $v_1 \neq 0$)
II. Eigenvectors

(NEED TO CHECK WHETHER A IS DIAGONALIZABLE OR NOT. LUCKILY, THERE ARE BUNCH OF USEFUL TESTS.) THE FIRST ONE CONCERNS THE CHARACTERISTIC POLYNOMIAL.

DEF. A POLYNOMIAL \( \phi(t) \) SPLIT OVER \( F \) IF

\[
\phi(t) = C(t-a_1) \cdots (t-a_n)
\]

For some \( a_i, C \in F \)

EX. \( \phi(t) = t^4 - 5t + 6 = (t-1)(t+3) \) SPLIT OVER \( \mathbb{R} \)

EX. \( \phi(t) = t^2 + 1 \) DOESN'T SPLIT OVER \( \mathbb{R} \), BUT SPLIT OVER \( \mathbb{C} \)

BECAUSE \( \phi(t) = (t-i)(t+i) \)

(IN FACT ANY POLY SPLIT OVER \( \mathbb{C} \), WHICH MAKES \( C \) SO NICE)

FACT. IF \( T \) IS DIAGONALIZABLE, THEN \( \phi(t) = \text{CHAR POLY OF } T \) MUST SPLIT.

WHY? BY ASSUMPTION, THERE IS A BASIS \( \beta \) SUCH THAT

\[
[T]_\beta \text{ IS DIAGONAL } = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}
\]

THEN \( \phi(t) = \text{DEF} \left( [T]_\beta^m - tI \right) \)

DEF

\[
= \begin{vmatrix} \lambda_1 - t \\ \vdots \\ \lambda_n - t \end{vmatrix}
\]

\[
= \begin{vmatrix} (\lambda_1 - t) \cdots (\lambda_n - t) \\ \vdots \\ - (t-\lambda_1) \cdots - (t-\lambda_n) \end{vmatrix}
\]

\[
= (-1)^n \cdot \prod_{i=1}^{n} (t-\lambda_i) \in \mathbb{R} \quad \text{or } \mathbb{C}
\]
**Test #1**

If \( f(t) \) does not split, then \( T \) (or \( A \)) is not diagonalizable!

\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

\[ f(t) = \begin{vmatrix} -t & -1 \\ 1 & -t \end{vmatrix} = t^2 + 1 \rightarrow \text{does not split (over } \mathbb{R}) \text{, so } A \text{ is not diagonalizable.} \]

**Note**

From now on, we assume \( f(t) \) splits.

**III - Eigenvalue Test**

(The next one concerns the eigenvalues of \( A \))

**Test #2**

If \( A \) (\( n \times n \)) has \( n \) distinct eigenvalues, then \( A \) is diagonalizable.

**Ex.**

\[ A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \]

has 2 eigenvalues: \( \lambda = 2, 3 \) (check)

so diagonalizable.

Why?

Let \( \lambda_1, \ldots, \lambda_n \) be the distinct eigenvalues of \( A \).

Let \( v_1, \ldots, v_n \) be the corresponding eigenvectors.

**By key lemma, \( \{v_1, \ldots, v_n\} \) is \( LT \), so a basis of \( \mathbb{F}^n \)

\( n \) vectors.

Hence get a basis of \( \mathbb{F}^n \) of eigenvectors of \( A \).

\[ \text{But } A \text{ could still be diagonalizable even if } A \text{ only has } \]

1 eigenvalue! (Ex. \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), \( \lambda = 1 \) but \( A \) diagonal.)
IV - EIGENVECTOR TEST

(If everything else fails, that is the char poly splits and we do not have distinct eigenvalues, then we really have to look at the eigenvectors.)

**Important Ex** is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ diagonalizable?

**Eigenvalues** $\lambda(t) = \text{det}(A - tI) = \begin{vmatrix} 1-t & 1 \\ 0 & 1-t \end{vmatrix} = (1-t)^2 = 0$

$\Rightarrow \lambda = 1$ with (algebraic) multiplicity $M_1 = 2$

**Eigenvectors** $\lambda = 1$: $\text{null}(A - I) = \text{null}\left[ \begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} = \text{null}\left[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] \right] = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(Eigenspace for $\lambda = 1$)

**Intuitively** not enough eigenvectors, so not diagonalizable

(How can you get a basis of $\mathbb{R}^2$ with just 1 eigenvector?)

**Diagonally** $\text{dim}(E_1) = 1 < 2 = M_1$ (multiplicity of $\lambda_1$)

So not diagonalizable

$\Rightarrow$ **Ultimate Test #3** suppose $A$ has eigenvalues $\lambda_1, \ldots, \lambda_k$ with multiplicities $M_1, \ldots, M_k$

Then $A$ is diagonalizable iff

"The eigenspace $E_i$ is as big as possible."

$\text{dim}(E_i) = M_i$ for all $i = 1, \ldots, k$