WEDNESDAY, JUNE 5, 2019

LECTURE 28 – DIAGONALIZABILITY (I) (SECTION 5.2)

PREVIOUSLY ON CAPTAIN PEYAMERICA, WE LEARNED ABOUT DIAGONALIZATION, WHICH IS A NEAT WAY OF TRANSFORMING A MATRIX INTO A DIAGONAL MATRIX.

RECALL A IS DIAGONALIZABLE $\Leftrightarrow$ THERE IS A BASIS OF $\mathbb{F}^N$ CONSISTING OF EIGENVECTORS OF A

MAIN Q HOW TO CONCRETELY CHECK IF A IS DIAGONALIZABLE

(NOTE: ALL THE RESULTS ARE TRUE BOTH FOR LT T AND MATRICES A, WILL FREQUENTLY SWITCH BETWEEN THE TWO)

I – KEY LEMMA

IT ALL RELIES ON THE FOLLOWING KEY LEMMA:

THEOREM IF $\lambda_1, \dots, \lambda_N$ ARE DISTINCT EIGENVALUES OF T AND $V_1, \dots, V_N$ ARE THE CORRESPONDING EIGENVECTORS, THEN $\{V_1, \dots, V_N\}$ IS LI

("EIGENVECTORS CORRESPONDING TO $\neq$ EIGENVALUES ARE LI")

EX $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\lambda_1 = -1 \rightsquigarrow \overset{V_1}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$, $\lambda_2 = 3 \rightsquigarrow \overset{V_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

THIS SAYS $\{ \underset{V_1}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}, \underset{V_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \}$ IS AUTOMATICALLY LI ($\Rightarrow$ A BASIS OF $\mathbb{R}^2$)

WHY? INDUCTION ON N

1) BASE $(N=1)$ $\{V_1\}$ IS LI (SINCE $V_1 \neq 0$)

2) IND SUPPOSE P_{N-1} IS TRUE, SHOW P_N IS TRUE

LET $V_1, \dots, V_N$ BE EIGENVECTORS CORRESP TO $\lambda_1, \dots, \lambda_N$ (DISTINCT)

AND SUPPOSE $a_1 V_1 + \cdots + a_N V_N = 0$ $(*)$

3) ON THE ONE HAND, APPLY T TO $(*)$:

$$T(a_1 v_1 + \cdots + a_N v_N) = T(\underline{0}) = \underline{0}$$

$$a_1 T(v_1) + \cdots + a_N T(v_N) = \underline{0}$$

$$a_1 \lambda_1 v_1 + \cdots + a_N \lambda_N v_N = \underline{0} \qquad (1)$$

4) ON THE OTHER HAND, MULTIPLY $(*)$ BY λ_1 (TRICK!)

$$\lambda_1 (a_1 v_1 + \cdots + a_N v_N) = \lambda_1 \underline{0}$$

$$a_1 \lambda_1 v_1 + \cdots + a_N \lambda_1 v_N = \underline{0} \qquad (2)$$

5) SUBTRACT THE TWO

$$\cancel{a_1 \lambda_1 v_1} + a_2 \lambda_2 v_2 + \cdots + a_N \lambda_N v_N = \underline{0}$$

$$-(\cancel{a_1 \lambda_1 v_1} + a_2 \lambda_1 v_2 + \cdots + a_N \lambda_1 v_N) = \underline{0}$$

$$a_2(\lambda_2 - \lambda_1) v_2 + \cdots + a_N(\lambda_N - \lambda_1) v_N = \underline{0}$$

6) BY INDUCTION HYP, $\underbrace{\{v_2, \ldots, v_N\}}_{N-1 \text{ VECTORS}}$ ARE LI, SO

SINCE $\lambda_1, \ldots, \lambda_N$ ARE DISTINCT

$$\left.\begin{array}{l} a_2 \overbrace{(\lambda_2 - \lambda_1)}^{\neq 0} = 0 \\ \vdots \\ a_N \underbrace{(\lambda_N - \lambda_1)}_{\neq 0} = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a_2 = 0 \\ \vdots \\ a_N = 0 \end{array}\right.$$

$\{v_1, \ldots, v_N\}$ LI ✓

$\Uparrow$

$\Rightarrow$ $(*)$ BECOMES $a_1 v_1 = \underline{0} \Rightarrow a_1 = 0$ (SINCE $v_1 \neq \underline{0}$)

II – ~~EIGENVALUES~~ CHAR POLY TEST

(RECALL THAT OUR GOAL IS TO CHECK WHETHER A IS DIAGONALIZABLE OR NOT. LUCKILY THERE ARE BUNCH OF USEFUL TESTS; THE FIRST ONE CONCERNS THE CHARACTERISTIC POLYNOMIAL)

DEF A POLYNOMIAL $f(t)$ SPLITS OVER $\mathbb{F}$ IF

$$f(t) = C(t-a_1)\cdots(t-a_N) \quad \text{FOR SOME } a_i, C \in \mathbb{F}$$

EX $f(t) = t^2 - 5t + 6 = (t-2)(t-3)$ SPLITS OVER $\mathbb{R}$

EX $f(t) = t^2 + 1$ DOESN'T SPLIT OVER $\mathbb{R}$, BUT SPLITS OVER $\mathbb{C}$ BECAUSE $f(t) = (t-i)(t+i)$

(IN FACT ANY POLY SPLITS OVER $\mathbb{C}$, WHICH MAKES $\mathbb{C}$ SO NICE)

FACT IF T IS DIAGONALIZABLE, THEN $f(t)$ (= CHAR POLY OF T) MUST SPLIT.

WHY? BY ASSUMPTION, THERE IS A BASIS β SUCH THAT

$$[T]_\beta^\beta \text{ IS DIAGONAL } = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$$

THEN $f(t) \underset{\text{DEF}}{=} \det\left([T]_\beta^\beta - tI\right)$

$$= \begin{vmatrix} \lambda_1 - t & & \\ & \ddots & \\ & & \lambda_N - t \end{vmatrix}$$

$$= (\lambda_1 - t)\cdots(\lambda_N - t)$$

$$= (-(t-\lambda_1))\cdots(-(t-\lambda_N)) = \underbrace{(-1)^N}_{C}\,(t-\underbrace{\lambda_1}_{a_1})\cdots(t-\underbrace{\lambda_N}_{a_N}) \checkmark$$

$\Rightarrow$ TEST #1 IF $f(t)$ DOESN'T SPLIT, THEN T (OR A) IS NOT DIAGONALIZABLE!

EX IS $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ DIAGONALIZABLE (OVER $\mathbb{R}$)?

$f(t) = \begin{vmatrix} -t & -1 \\ 1 & -t \end{vmatrix} = t^2+1 \rightarrow$ DOESN'T SPLIT (OVER $\mathbb{R}$), SO NO

(BUT DIAGONALIZABLE OVER $\mathbb{C}$, SO THIS IS THE RARE CASE WHERE THE FIELD MATTERS)

NOTE FROM NOW ON, WE ASSUME $f(t)$ SPLITS

III - EIGENVALUE TEST

(THE NEXT ONE CONCERNS THE EIGENVALUES OF A)

TEST #2 IF A $(N \times N)$ HAS N DISTINCT EIGENVALUES, THEN A IS DIAGONALIZABLE.

EX $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ HAS 2 EIGENVALUES: $\lambda = 2, 3$ (CHECK)

SO DIAGONALIZABLE

WHY? LET $\lambda_1, \dots, \lambda_N$ BE THE DISTINCT EIGENVALUES OF A

LET $v_1, \dots, v_N$ BE THE CORRESPONDING EIGENVECTORS

BY KEY LEMMA, $\underbrace{\{v_1, \dots, v_N\}}_{N \text{ VECTORS}}$ IS LI, SO A BASIS OF $\mathbb{F}^N$

HENCE GET A BASIS OF $\mathbb{F}^N$ OF EIGENVECTORS OF A ■

⚠ BUT A COULD STILL BE DIAGONALIZABLE EVEN IF A ONLY HAS 1 EIGENVALUE! (EX $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda = 1$ BUT A DIAG.)

IV – EIGENVECTOR TEST

(IF EVERYTHING ELSE FAILS, THAT IS THE CHAR POLY SPLITS AND WE DO NOT HAVE DISTINCT EIGENVALUES, THEN WE REALLY HAVE TO LOOK AT THE EIGENVECTORS)

IMPORTANT EX IS $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ DIAGONALIZABLE?

EIGENVALUES $f(t) = \det(A - tI) = \begin{vmatrix} 1-t & 1 \\ 0 & 1-t \end{vmatrix} = (1-t)^{2} = 0$

$\Rightarrow \lambda = 1$ WITH (ALGEBRAIC) MULTIPLICITY $M_1 = 2$

EIGENVECTORS $\lambda = 1$ $\underbrace{Nul(A - I)}_{E_1} = Nul \begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} = Nul \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(EIGENSPACE FOR $\lambda = 1$) $= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

INTUITIVELY: NOT ENOUGH EIGENVECTORS, SO NOT DIAGONALIZABLE (B/C HOW CAN YOU GET A BASIS OF $\mathbb{R}^2$ WITH JUST 1 EIGEN-VECTOR?)

RIGOROUSLY: $\dim(E_1) = 1 < 2 = M_1$ (MULTIPLICITY OF λ_1)

SO NOT DIAGONALIZABLE

$\Rightarrow$ ULTIMATE TEST # 3 SUPPOSE A HAS EIGENVALUES $\lambda_1, \dots, \lambda_k$ WITH MULTIPLICITIES $M_1, \dots, M_k$

THEN A IS DIAGONALIZABLE IFF

"THE EIGENSPACE IS AS BIG AS POSSIBLE" $\longrightarrow$ $\boxed{\dim(E_{\lambda_i}) = M_i}$ FOR ALL $i = 1, \dots, k$