FRIDAY, JUNE 7, 2019

LECTURE 29 – DIAGONALIZABILITY (II) (SECTION 5.2)

THERE'S A SAYING IN GERMAN THAT SAYS "EVERYTHING HAS AN END, EXCEPT FOR A SAUSAGE, WHICH HAS TWO" AND WITH THIS, I WOULD LIKE TO WELCOME YOU TO THE FINAL LECTURE OF MATH 121A AND THE SECOND PART OF OUR DIAGONALIZABILITY ADVENTURE!

MAIN Q HOW TO CHECK IF A (OR T) IS DIAGONALIZABLE?

I – MULTIPLICITY AND EIGENSPACE

(LET ME FIRST RECALL SOME NOTATION FROM LAST TIME)

DEF SUPPOSE $f(t) = c\,(t-a_1)^{m_1} \cdots (t-a_k)^{m_k}\, g(t)$

WHERE THE a_i ARE DISTINCT AND g HAS NO ROOTS,

THEN WE SAY $t = a_i$ HAS MULTIPLICITY m_i

EX $f(t) = (t-2)^3 (t-4)^5 (t-6)$ (SPLITS)

THEN $t = 2$ HAS MULTIPLICITY 3 $(m_1 = 3)$
$t = 4$ " 5 $(m_2 = 5)$
$t = 6$ " 1 $(m_3 = 1)$

NOTE $m_1 + m_2 + m_3 = 3 + 5 + 1 = 9 = \deg(f)$

FACT IF f SPLITS, THEN $m_1 + \cdots + m_k = n = \deg(f)$

DEF IF λ IS AN EIGENVALUE OF A, THEN

$E_\lambda = \mathrm{Nul}(A - \lambda I) = \{x \mid Ax = \lambda x\}$ IS THE EIGENSPACE OF A CORRESP TO λ

II – ULTIMATE DIAGONALIZATION TEST

TEST #3 IF T (OR A) HAS EIGENVALUES $\lambda_1, \dots, \lambda_k$ WITH CORRESPONDING MULTIPLICITIES $m_1, \dots, m_k$, THEN:

$$T \text{ IS DIAGONALIZABLE} \iff \dim(E_{\lambda_i}) = m_i \text{ FOR ALL } i = 1, \dots, k$$

"E_{λ_i} AS BIG AS POSSIBLE"

EX IS $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ DIAGONALIZABLE?

$$f(t) = -(t-3)^2(t-2) = 0 \quad \text{(CHECK)}$$

$\Rightarrow$ EIGENVALUES $\lambda_1 = 3 \ (m_1 = 2)$, $\lambda_2 = 2 \ (m_2 = 1)$

EIGENVECTORS $\lambda_1 = 3$

$$E_3 = \text{NUL}(A - 3I) = \text{NUL}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \text{NUL}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2 PIVOTS, SO $3 - 2 = 1$ FREE VAR

HENCE $\dim(E_3) = 1 < m_1 = 2$, SO NOT DIAGONALIZABLE!

NOTE 1) NO NEED TO FIND A BASIS OF E_3
2) NO NEED TO FIND E_2 (B/C E_3 IS ALREADY TOO SMALL)

~~III – "PAGE" OF (??)~~

EX Is $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ DIAGONALIZABLE?

CHECK $f(t) = -(t-1)^2(t-2) = 0$

$\Rightarrow \lambda = 1 \ (m_1 = 2), \ \lambda = 2 \ (m_2 = 1)$

$E_1 = \mathrm{Nul}(A - I) = \mathrm{Nul} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \mathrm{Nul} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1 PIVOT $\Rightarrow$ 2 FREE VAR $\Rightarrow$ $\dim(E_1) = 2 = m_1$ ✓

E_2: SIMILARLY CHECK $\dim(E_2) = 1 = m_2$ ✓

HENCE A IS DIAGONALIZABLE ✓

WHY THIS WORKS?

CAN CHECK THAT $\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ IS A BASIS OF E_1

$\beta_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ IS A BASIS OF E_2

TURNS OUT $\beta_1 \cup \beta_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ IS A BASIS OF $\mathbb{F}^3$

SO GET A BASIS OF EIGENVECTORS

(NEED $\dim(E_{\lambda_i}) = m_i$ TO GUARANTEE THAT WE HAVE ENOUGH EIGEN-VECTORS).

III — PROOF

($\Rightarrow$) SKIP (AIN'T NOBODY GOT TIME FOR THAT !!)

LEMMA: IF $\lambda_1, \dots, \lambda_k$ ARE DISTINCT AND $v_i \in E_{\lambda_i}$ FOR $i = 1, \dots, k$

THEN $v_1 + \dots + v_k = \underline{0} \Rightarrow v_1 = \underline{0}, \dots, v_k = \underline{0}$

(SORT OF AN ANALOG OF LI; PROOF USES KEY LEMMA FROM LAST TIME)

PROOF OF ($\Leftarrow$) $\quad \dim(E_{\lambda_i}) = m_i \quad (i = 1, \dots, k) \Rightarrow T$ DIAGONALIZABLE

SUPPOSE FOR SIMPLICITY $k=2$ (GENERAL CASE IS SIMILAR)

SO T HAS EIGENVALUES λ_1 & λ_2

LET $\beta_1 = \{v_1, \dots, v_{m_1}\}$ BE A BASIS OF E_{λ_1} (SINCE $\dim(E_{\lambda_1}) = m_1$)

$\beta_2 = \{w_1, \dots, w_{m_2}\}$ // E_{λ_2} (SINCE $\dim(E_{\lambda_2}) = m_2$)

CLAIM $\beta = \beta_1 \cup \beta_2$ IS A BASIS OF V

(SO GET A BASIS OF V OF EIGENVECTORS OF T ✓)

NOTE β HAS $m_1 + m_2 = n$ VECTORS, SO ENOUGH TO CHECK β IS LI

DEG(f)

SPS $a_1 v_1 + \dots + a_{m_1} v_{m_1} + b_1 w_1 + \dots + b_{m_2} w_{m_2} = \underline{0} \quad (*)$

LET $v = a_1 v_1 + \dots + a_{m_1} v_{m_1} \in E_{\lambda_1}$ SINCE $v_i \in E_{\lambda_1}$

$w = b_1 w_1 + \dots + b_{m_2} w_{m_2} \in E_{\lambda_2}$

THEN $(*)$ SAYS $v + w = \underline{0}$, SO BY LEMMA $v = \underline{0}$ & $w = \underline{0}$

SO $v = a_1 v_1 + \dots + a_{m_1} v_{m_1} = \underline{0} \Rightarrow a_1 = 0, \dots, a_{m_1} = 0$ (β_1 BASIS)

$w = b_1 w_1 + \dots + b_{m_2} w_{m_2} = \underline{0} \Rightarrow b_1 = 0, \dots, b_{m_2} = 0$ (β_2 BASIS)

HENCE β IS LI ∎

IV – GRAND FINALE

SINCE I WANT TO END THIS SCHOOL YEAR ON A POSITIVE NOTE, AND NOT ON A DRY PROOF, LET ME FINISH WITH A COOL EXAMPLE!

EX FIND $\sqrt{A}$, $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

1) DIAGONALIZE A: $\lambda = 1 \rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A = PDP^{-1}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

2) MAIN OBSERVATION

IF $A = PDP^{-1}$, $A^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$

AND SIMILARLY $A^N = PD^NP^{-1}$, $D^N = \begin{bmatrix} 1^N & 0 \\ 0 & 4^N \end{bmatrix}$

3) AMAZING FACT THIS WORKS NOT ONLY FOR INTEGER N, BUT REALLY ANY N YOU WANT!

IN PARTICULAR: $\sqrt{A} = A^{1/2}$

$$= PD^{1/2}P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 4^{1/2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\sqrt{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, CHECK $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = A$!!!