SOLUTIONS

1. *(15 = 2 + 1 + 12 points)* Let V be a finite-dimensional vector space

(a) Define: $\dim(V)$

(b) What theorem is used to show that $\dim(V)$ is well-defined? Just tell me the name of the theorem.

(c) Let W and Z be two subspaces of V such that $W \cap Z = \{0\}$.

Show that $\dim(W + Z) = \dim(W) + \dim(Z)$

Note: $W + Z$ means $\{\mathbf{w} + \mathbf{z} \mid \mathbf{w} \in W \text{ and } \mathbf{z} \in Z\}$

Hint: Start with a basis of W and a basis of Z

(a) DIM(V) = # OF VECTORS IN ANY BASIS OF V

(b) REPLACEMENT THEOREM

(c) LET $\beta_1 = \{w_1, \dots, w_m\}$ BE A BASIS OF W ($m = \dim(W)$)
AND $\beta_2 = \{z_1, \dots, z_k\}$ BE A BASIS OF Z ($k = \dim(Z)$)

CLAIM $\beta = \{w_1, \dots, w_m, z_1, \dots, z_k\}$ IS A BASIS OF $W + Z$

1) SPANS LET $v \in W + Z$, THEN $v = w + z$ WHERE $w \in W$ & $z \in Z$
SINCE β_1 IS A BASIS OF W, $w = a_1 w_1 + \dots + a_m w_m$ FOR SOME $a_i \in F$
SINCE β_2 IS A BASIS OF Z, $z = b_1 z_1 + \dots + b_k z_k$ FOR SOME $b_j \in F$
THEN $v = w + z = a_1 w_1 + \dots + a_m w_m + b_1 z_1 + \dots + b_k z_k \in \text{SPAN}(\beta)$ ✓

2) LI SUPPOSE $a_1 w_1 + \dots + a_m w_m + b_1 z_1 + \dots + b_k z_k = \underline{0}$ (*)
THEN $a_1 w_1 + \dots + a_m w_m = -b_1 z_1 \dots - b_k z_k \in Z$ (SINCE Z IS A SUBSPACE)
BUT ALSO $a_1 w_1 + \dots + a_m w_m \in W$ (SINCE W IS A SUBSPACE)
HENCE $a_1 w_1 + \dots + a_m w_m \in W \cap Z = \{\underline{0}\}$
SO $a_1 w_1 + \dots + a_m w_m = \underline{0}$ AND SO $a_1 = 0, \dots, a_m = 0$ SINCE $\beta_1 = \{w_1, \dots, w_m\}$ IS LI
HENCE (*) BECOMES $b_1 z_1 + \dots + b_k z_k = \underline{0}$, SO $b_1 = 0, \dots, b_k = 0$
SINCE $\beta_2 = \{z_1, \dots, z_k\}$ IS LI
HENCE $a_1 = 0, \dots, a_m = 0$, $b_1 = 0, \dots, b_k = 0$ ✓

THEREFORE $\dim(W + Z) = |\beta| = m + k = \dim(W) + \dim(Z)$ ∎

2. *(15 = 2 + 10 + 3 points)* Let V be a finite-dimensional vector space and suppose $T : V \to V$ is linear.

   (a) Define: $rank(T)$

   (b) Define $W = \{v \in V \mid T(v) = v\}$. Let $k = \dim(W)$ and assume $W \neq \{0\}$. Show that there is a β of V and matrices B and C such that

$$[T]_\beta^\beta = \begin{bmatrix} I_k & B \\ O & C \end{bmatrix}$$

   Hint: Start with a basis of W.

   (c) With the notation as above, find an identity relating $rank(T)$, $\dim(W)$ and $rank(C)$. No proof required.

(a) $\text{RANK}(T) = \text{DIM}(R(T))$

(b) LET $\{V_1, \cdots, V_K\}$ BE A BASIS OF W, AND EXTEND THIS TO A BASIS $\{V_1, \cdots, V_K, V_{K+1}, \cdots, V_N\}$ OF V $(N = \text{DIM}(V))$

THEN
$$[T]_\beta^\beta = \begin{array}{c} V_1 \\ \vdots \\ V_K \\ \vdots \\ V_N \end{array} \begin{bmatrix} 1 & & & 0 & A_{1K+1} & \cdots & A_{1N} \\ 0 & 1 & & 0 & \vdots & & \vdots \\ 0 & & 1 & & & & \\ 0 & & & 1 & A_{KK+1} & \cdots & A_{KN} \\ \vdots & & 0 & 0 & A_{K+1K+1} & & A_{K+1N} \\ 0 & & & 0 & A_{NK+1} & & A_{NN} \end{bmatrix}$$
$$\qquad\qquad T(V_1) \quad \cdots \quad T(V_K) \quad T(V_{K+1}) \quad \cdots \quad T(V_N)$$

NOW FOR $j = 1, \cdots, K$, SINCE $V_j \in W$,

$T(V_j) = V_j = \underline{\underline{0}}\, V_1 + \cdots + \underline{\underline{1}}\, V_j + \cdots + \underline{\underline{0}}\, V_N$ (WHICH IS 1 PRECISELY IN THE j^{TH} ENTRY)

AND FOR $j = K+1, \cdots, N$, $T(V_j) = A_{1j} V_1 + \cdots + A_{Nj} V_N$ FOR SOME SCALARS A_{ij}

HENCE IF YOU LET B BE THE MATRIX WITH ENTRIES $\begin{bmatrix} A_{1K+1} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{KK+1} & \cdots & A_{KN} \end{bmatrix}$

AND C BE THE MATRIX WITH ENTRIES $\begin{bmatrix} A_{K+1K+1} & \cdots & A_{K+1N} \\ \vdots & & \vdots \\ A_{NK+1} & \cdots & A_{NN} \end{bmatrix}$

THEN $[T]_\beta^\beta = \begin{bmatrix} I_K & B \\ 0 & C \end{bmatrix}$

(c) $\text{RANK}(T) = \text{DIM}(W) + \text{RANK}(C)$

3. *(15 = 12 + 3 points)*

(a) Show that, for A below, we have

$$\det(A + tI) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix}$$

(b) Use (a) to find the characteristic polynomial of A

(a) BY INDUCTION ON N

LET P_N BE THE PROPOSITION ABOVE

BASE CASE $(N=2)$ THEN $\text{DET}(A+tI) = \begin{vmatrix} t & a_0 \\ -1 & t+a_1 \end{vmatrix} = t(t+a_1)+a_0$
$= t^2 + ta_1 + a_0$
$= a_0 + a_1 t + t^2$ ✓

IND STEP SUPPOSE P_{N-1} IS TRUE, SHOW P_N IS TRUE

$$\text{DET}(A+tI) = \begin{vmatrix} t & 0 & \cdots & a_0 \\ -1 & t & \cdots & a_1 \\ & & \ddots & \\ & & -1 & t+a_{N-1} \end{vmatrix} = t \begin{vmatrix} t & \cdots & a_1 \\ -1 & t \cdots & a_2 \\ & \ddots & \\ & -1 & t+a_{N-1} \end{vmatrix} + (-1)^{1+N} a_0 \begin{vmatrix} -1 & t \cdots & a_1 \\ & \ddots & \\ & & -1 \end{vmatrix}$$

INDUCTION HYPOTHESIS

$= t\left(a_1 + a_2 t + \cdots + a_{N-1} t^{N-2} + t^{N-1}\right) + (-1)^{\cancel{1}+N} a_0 (-1)^{N-\cancel{1}}$ (UPPER TRIANGULAR MATRIX)

$= a_1 t + a_2 t^2 + \cdots + a_{N-1} t^{N-1} + t^N + \underbrace{(-1)^{2N}}_{1} a_0$

$= a_0 + a_1 t + a_2 t^2 + \cdots + a_{N-1} t^{N-1} + t^N$ ✓

HENCE P_N IS TRUE, SO P_N IS TRUE FOR ALL N.

(b) CHAR POLY $= \text{DET}(A - tI) = \text{DET}(A + (-t)I)$

$\overset{(a)}{=} a_0 + a_1(-t) + \cdots + a_{N-1}(-t)^{N-1} + (-t)^N$

$= a_0 + a_1(-1)t + \cdots + a_{N-1}(-1)^{N-1} t^{N-1} + (-1)^N t^N$

4. *(10 points)* Let V be a finite-dimensional vector space and suppose $T : V \to V$ is linear. Show that (nonzero) eigenvectors of T corresponding to distinct eigenvalues are linearly independent.

BY INDUCTION ON N = NUMBER OF DISTINCT EIGENVALUES OF T

LET P_N BE THE PROPOSITION: IF $\lambda_1, \dots, \lambda_N$ ARE DISTINCT EIGENVALUES OF T AND $v_1, \dots, v_N$ ARE THE CORRESPONDING EIGENVECTORS, THEN $\{v_1, \dots, v_N\}$ IS LI

BASE CASE ($N=1$) IF T ONLY HAS 1 EIGENVALUE λ_1 AND IF v_1 IS A CORRESPONDING EIGENVECTOR, THEN $\{v_1\}$ IS LI SINCE $v_1 \neq \underline{0}$

IND STEP SUPPOSE P_{N-1} IS TRUE, SUPPOSE P_N IS TRUE

LET $\lambda_1, \dots, \lambda_N$ BE DISTINCT EIGENVALUES OF T AND $v_1, \dots, v_N$ BE CORRESPONDING EIGENVECTORS, AND SUPPOSE

$$a_1 v_1 + \dots + a_N v_N = \underline{0} \quad (*)$$

) ON THE ONE HAND, APPLY T TO $(*)$ TO GET:

$$T(a_1 v_1 + \dots + a_N v_N) = T(\underline{0})$$

$$a_1 T(v_1) + \dots + a_N T(v_N) = \underline{0}$$

$$a_1 \lambda_1 v_1 + \dots + a_N \lambda_N v_N = \underline{0} \quad (1)$$

) ON THE OTHER HAND, MULTIPLY $(*)$ BY λ_N TO GET:

$$\lambda_N (a_1 v_1 + \dots + a_N v_N) = \lambda_N (\underline{0})$$

$$a_1 \lambda_N v_1 + \dots + a_N \lambda_N v_N = \underline{0} \quad (2)$$

) SUBTRACT (2) FROM (1):

$(1)-(2)$: $a_1(\lambda_1 - \lambda_N) v_1 + \dots + a_{N-1}(\lambda_{N-1} - \lambda_N) v_{N-1} = \underline{0}$

BUT BY INDUCTIVE HYPOTHESIS, $\{v_1, \dots, v_{N-1}\}$ IS LI, SO

$$\begin{cases} a_1(\lambda_1 - \lambda_N) = 0 \\ \vdots \\ a_{N-1}(\lambda_{N-1} - \lambda_N) = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ \vdots \\ a_{N-1} = 0 \end{cases}$$

SINCE $\lambda_1, \dots, \lambda_N$ ARE DISTINCT

SO $(*)$ BECOMES $a_N \underbrace{v_N}_{\neq 0} = \underline{0}$ SO $a_N = 0$, SO $a_1 = \dots = a_N = 0$, HENCE $\{v_1, \dots, v_N\}$ IS LI ✓ SO P_N IS TRUE

5. *(15 = 7 + 8 points)* Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A^T$

(a) Calculate $A = [T]_\beta^\beta$, where β is the standard basis of $M_{2\times 2}$:

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(b) Is T diagonalizable? Why or why not?

(a) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \underline{\underline{1}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \underline{\underline{0}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \underline{\underline{1}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \underline{\underline{0}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \underline{\underline{1}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{0}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \underline{\underline{0}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \underline{\underline{1}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

SO $A = [T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) <u>EIGENVALUES</u> $\quad \det(A - tI) = \begin{vmatrix} 1-t & 0 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 1-t \end{vmatrix} = (1-t)^2 \begin{vmatrix} -t & 1 \\ 1 & -t \end{vmatrix}$

$= (t-1)^2 (t^2 - 1) = (t-1)^2 (t-1)(t+1) = (t-1)^3 (t+1) = 0$

<u>EIGENVALUES</u> $\lambda_1 = 1 \quad (M_1 = 3)$, $\lambda_2 = -1 \quad (M_2 = 1)$

<u>EIGENSPACES</u> $E_1 = \mathrm{NUL}(A - I) = \mathrm{NUL} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{NUL} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1 PIVOT, SO 3 FREE VAR, SO $\dim(E_1) = \dim(\mathrm{NUL}(A - I)) = 3 = M_1$ ✓

E_{-1} : WE KNOW $E_{-1} \neq \{0\}$, so $\dim(E_{-1}) \geq 1$,

ALSO $\dim(E_{-1}) \leq M_2 = 1$, SO $\dim(E_{-1}) = 1 = M_2$ ✓

SINCE $\dim(E_{\lambda_i}) = M_i$ FOR ALL i,

A (AND HENCE T) IS DIAGONALIZABLE ●

6. *(15 points, 3 points each)* Mark each of the following statements as True or False. *Briefly* justify your answers

FALSE

(a) Let $\beta = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis of V with dual basis $\beta^* = \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}\}$. Then $\{\mathbf{f_1}, 2\mathbf{f_2}, 3\mathbf{f_3}\}$ is the dual basis of $\{\mathbf{v_1}, 2\mathbf{v_2}, 3\mathbf{v_3}\}$

$2f_2(2V_2)$ SHOULD BE 1, BUT

$2f_2(2V_2) = 4f_2(V_2) = 4(1) = 4$

FALSE

(b) If A is invertible, then A is diagonalizable

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ IS INVERTIBLE SINCE $\det(A) = 1$,

BUT A IS NOT DIAGONALIZABLE (FROM CLASS)

FALSE

(c) If $\{\mathbf{w_1}, \cdots, \mathbf{w_n}\}$ is a basis of W and $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ is subset of V then there exists $T : V \to W$ linear such that $T(\mathbf{v_i}) = \mathbf{w_i}$ for all $i = 1, \cdots, n$.

LET $\{V_1, V_2\} = \{(1,0), (2,0)\}$ AND $\{W_1, W_2\} = \{(1,0), (0,1)\}$

THEN $T(V_1) = W_1 \Rightarrow T(1,0) = (1,0)$

BUT $T(V_2) = W_2 \Rightarrow T(2,0) = (0,1)$, BUT $T(2,0) = 2T(1,0)$
$= 2(1,0) = (2,0)$ $\Rightarrow\Leftarrow$

FALSE

(d) The function $T : M_{2\times 2} \to M_{2\times 2}$ defined by $T(A) = \det(A)$ is a linear transformation

$\det\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4$

$\neq$

$2\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2(1) = 2$

TRUE

(e) If you row-reduce a matrix A to get a matrix B, then $rank(A) = rank(B)$.

KNOW $CA = B$ FOR SOME INVERTIBLE C
(= PRODUCT OF ELEM MATRICES)

SO $\mathrm{RANK}(\underset{\text{INV}}{C}A) = \mathrm{RANK}(B)$

$\mathrm{RANK}(A) = \mathrm{RANK}(B)$

7. *(15 = 9 + 3 + 3 points)* Let $V = P_n(\mathbb{R})$ and $c_0, c_1, \cdots, c_n$ be distinct real numbers.

(a) For $i = 0, \cdots, n$, define $f_i \in V^*$ by $f_i(p) = p(c_i)$.

Show that $\gamma = \{f_0, \cdots, f_n\}$ is a basis of V^*.

Hint: Apply your equation to $p = \cancel{(x-c_1)(x-c_2)\cdots(x-c_n)}$. $(x-c_0)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)$

(b) Deduce from (a) that there is a basis of $\beta = \{p_0, \cdots, p_n\}$ of V such that for all $i, j = 0, \cdots, n$

$$p_i(c_j) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases}$$

(c) Conclude from (b) that for any polynomial $p \in V$, we have

$$p(x) = \sum_{i=0}^{n} p(c_i)p_i(x)$$

(a) SINCE $\dim(V^*) = \dim(V) = N+1$ AND γ HAS $N+1$ ELEMENTS, ENOUGH TO SHOW γ IS LI

SUPPOSE $a_0 f_0 + \cdots + a_N f_N = \underline{0} \leftarrow$ 0 FUNCTIONAL

THEN $(a_0 f_0 + \cdots + a_N f_N)(p) = \underline{0}(p)$

$a_0 f_0(p) + \cdots + a_N f_N(p) = 0$

$a_0 p(c_0) + \cdots + a_N p(c_N) = 0 \quad (*)$

FOR EACH $i = 0, \cdots, N$, APPLY $(*)$ TO $p = (x-c_0)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_N)$

TO GET: $a_0 \underbrace{\cancel{(c_0-c_0)}}_{0}\cdots(c_0-c_{i-1})(c_0-c_{i+1})\cdots(c_0-c_N)$

$+ \cdots + a_i(c_i-c_0)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_N)$

$+ \cdots + a_N(c_N-c_0)\cdots(c_N-c_{i-1})(c_N-c_{i+1})\cdots\underbrace{\cancel{(c_N-c_N)}}_{0} = 0$

$\Rightarrow a_i \underbrace{(c_i-c_0)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_N)}_{\neq 0 \text{ SINCE } c_0, \cdots, c_N \text{ ARE DISTINCT}} = 0$

$\Rightarrow \underline{a_i = 0}$, AND SINCE i IS ARBITRARY,

WE GET $a_0 = a_1 = \cdots = a_N = 0$ ✓

(b) FROM CLASS, WE KNOW $\gamma = \beta^*$ FOR SOME BASIS $\beta = \{p_0, \dots, p_N\}$ OF V.

SINCE $\gamma = \{f_0, \dots, f_N\}$ IS THE DUAL BASIS OF $\beta = \{p_0, \dots, p_N\}$

FOR ALL i, j $\quad f_j(p_i) = \begin{cases} 1 & \text{IF } i = j \quad (\text{THAT IS, } j = i) \\ 0 & \text{IF } i \neq j \quad (\text{THAT IS, } j \neq i) \end{cases}$

THAT IS, BY DEF OF p_i, $\quad p_i(c_j) = \begin{cases} 1 & \text{IF } j = i \\ 0 & \text{IF } j \neq i \end{cases}$ ✓

(c) IF $p \in V$, SINCE $\beta = \{p_0, \dots, p_N\}$ IS A BASIS OF V,

$p = a_0 p_0 + \dots + a_N p_N$ FOR SOME CONSTANTS $a_0, \dots, a_N$

THAT IS, FOR ALL x, $\quad p(x) = a_0 p_0(x) + \dots + a_N p_N(x)$

NOW FOR $j = 0, \dots, N$, LET $x = c_j$ TO GET:

$$p(c_j) = a_0 \cancel{p_0(c_j)}^{0} + \dots + a_j \underbrace{p_j(c_j)}_{1} + \dots + a_N \cancel{p_N(c_j)}^{0} \quad (\text{BY (b)})$$

$\Rightarrow p(c_j) = a_j$, SO $\underline{a_j = p(c_j)}$.

HENCE $\quad p(x) = p(c_0) p_0(x) + \dots + p(c_N) p_N(x) = \sum_{i=1}^{N} p(c_i) p_i(x)$

<u>OMG WAY</u> $\quad$ <u>FIX $x \in \mathbb{R}$</u> AND DEFINE $f \in V^*$ BY $f(p) = p(x)$

THEN SINCE $\gamma = \{f_0, \dots, f_N\}$ IS THE DUAL BASIS OF $\beta = \{p_0, \dots, p_N\}$.

WE HAVE $\quad f = \sum_{i=0}^{N} f(p_i) f_i = \sum_{i=0}^{N} p_i(x) f_i \quad$ (DEF OF f)

SO FOR EVERY $p \in V$,

$$f(p) = \left(\sum_{i=0}^{N} p_i(x) f_i \right) p = \sum_{i=0}^{N} p_i(x) f_i(p) = \sum_{i=0}^{N} p_i(x) p(c_i)$$

DEF OF f

$$p(x) = \sum_{i=0}^{N} p(c_i) p_i(x) \quad \blacksquare$$