MATH 121A – FINAL EXAM REVIEW SESSION

Note: Most of those problems are taken from the book, and you can refer to the solutions on the following webpage: Solutions.

Problem 1: Let V be the space of sequences with values in $\mathbb{R}$, and define $L : V \to V$ and $R : V \to V$ (the left-shift and right-shift transformations) by:

$$L(a_1, a_2, a_3, \cdots) = (a_2, a_3, \cdots) \qquad R(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots)$$

(a) Show that L is onto, but not one-to-one, and R is one-to-one but not onto
(b) Show that $LR = I$ but $RL \neq I$
(c) Find the eigenvalues of L and the eigenvalues of R (if they exist)

Note: Here you need to use the definition of eigenvalues, since we're dealing with an infinite dimensional vector space

Solution: Oh Shift!

Problem 2: (24 in 4.3) Show by induction on n that $\det(A + tI) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$, where

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix}$$

Solution: Neat Determinant
Problem 3: (Theorem 5.11 on page 278)

Definition: Let $W_1, \cdots, W_k$ be subspaces of V. Then $V = W_1 \oplus \cdots \oplus W_k$ iff the following two conditions are satisfied:

(1) $V = W_1 + \cdots + W_k$

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(2) If $w_1 + \cdots + w_k = 0$ (where $w_i \in W_i$), then $w_1 = \cdots = w_k = 0$

Suppose $T : V \to V$ has eigenvalues $\lambda_1, \cdots, \lambda_k$ and $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ Show that T is diagonalizable.

Hint: For each $i = 1, \cdots, k$, let β_i be a basis for E_{λ_i}, then show $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis of V. This is very similar to the proof given in Lecture 29.

Solution: Diagonalization and Direct Sums (the first half only)

Problem 4: (20a in 2.6) Suppose V and W are finite-dimensional, and suppose $T : V \to W$. Show that T is onto if and only if T^T (the transpose of T) is one-to-one.

Hint: First show that if W' is a proper subspace of W, $f \in W^\star$ with f nonzero but $f(x) = 0$ for all $x \in W'$.

Solution: One-to-one iff onto

Problem 5: (21 in 4.3) Show that if $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{bmatrix} A & B \\ O & C \end{bmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A)\det(C)$.

Hint: First show that $\det(M)$ is zero if C is not invertible (argue in terms of pivots), then prove the result holds if $A = I$ (do induction on the size of I), and you can similarly assume the result holds for $C = I$. Finally use

$$\begin{bmatrix} A & B \\ O & C \end{bmatrix} = \begin{bmatrix} I & O \\ O & C^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ O & C \end{bmatrix}$$

Solution: Determinant of a Block Matrix

Problem 6: (29a in 1.6) Suppose V is finite-dimensional and W_1 and W_2 are subspaces of V. Show

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Hint: Start with a basis β of $W_1 \cap W_2$. On the one hand, extend extend β to a basis β_1 of W_1. On the other hand, extend β to a basis β_2 of W_2. Show

$\beta_1 \cup \beta_2$ is a basis of $W_1 + W_2$.

Solution: $\dim(W_1 + W_2)$

Problem 7: (35 in 2.1) Suppose V is finite-dimensional and $T : V \to V$. Show:

$$V = R(T) + N(T) \Leftrightarrow R(T) \cap N(T) = \{0\}$$

Hint: Use Problem 6

Problem 8: (16 in 2.2) Suppose V and W are finite-dimensional with $\dim(V) = \dim(W)$, and suppose $T : V \to W$ is linear. Show there there exist ordered bases β and γ of V and W respectively such that $[T]_\beta^\gamma$ is diagonal.

Hint: Start with a basis of $N(T)$ and mimic the proof of the rank-nullity theorem to find a basis of $R(T)$, and then extend it to a basis γ of W.

Solution: Pseudo Diagonalization

Problem 9: Let $S = \{v_1, \cdots v_n\}$ be a subset of V (not necessarily a basis), and define $T : \mathbb{F}^n \to V$ by:

$$T(a_1, \cdots, a_n) = a_1 v_1 + \cdots + a_n v_n$$

(a) Show T is one-to-one if and only if S is linearly independent
(b) Show T is onto V if and only if S spans V

Solution: One-to-one iff linearly independent

Problem 10: (3 in 3.4) Show that $Ax = b$ is inconsistent if and only if the last column of $[A|b]$ is a pivot column

Hint: Use the rank criterion (Theorem 3.11) and remember that the rank is equal to the number of pivots

Solution: Rank Criterion Consequences