MATH 121A – FINAL EXAM

Name: ________________________

Student ID: ________________________

Instructions: This is it, your final hurdle to freedom!!! You have 120 minutes to take this exam, for a total of 100 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your answer, but also on your work, so please write in complete sentences and explain your steps as much as you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. **You will lose 2 points if you don't sign the statement below**. May your luck be diagonalizable!

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: ________________________________

1		15
2		15
3		15
4		10
5		15
6		15
7		15
Total		100

Date: Tuesday, June 11, 2019.

1. *(15 = 2 + 1 + 12 points)* Let V be a finite-dimensional vector space

   (a) Define: $\dim(V)$

   (b) What theorem is used to show that $\dim(V)$ is well-defined? Just tell me the name of the theorem.

   (c) Let W and Z be two subspaces of V such that $W \cap Z = \{\mathbf{0}\}$.

   Show that $\dim(W + Z) = \dim(W) + \dim(Z)$

   Note: $W + Z$ means $\{\mathbf{w} + \mathbf{z} \mid \mathbf{w} \in W \text{ and } \mathbf{z} \in Z\}$

   Hint: Start with a basis of W and a basis of Z

2. *(15 = 2 + 10 + 3 points)* Let V be a finite-dimensional vector space and suppose $T : V \to V$ is linear.

   (a) Define: $rank(T)$

   (b) Define $W = \{v \in V \mid T(v) = v\}$. Let $k = \dim(W)$ and assume $W \neq \{\mathbf{0}\}$. Show that there is a β of V and matrices B and C such that

   $$[T]_\beta^\beta = \begin{bmatrix} I_k & B \\ O & C \end{bmatrix}$$

   Hint: Start with a basis of W.

   (c) With the notation as above, find an identity relating $rank(T)$, $\dim(W)$ and $rank(C)$. No proof required.

3. *(15 = 12 + 3 points)*

(a) Show that, for A below, we have

$$\det(A + tI) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix}$$

(b) Use (a) to find the characteristic polynomial of A

4. *(10 points)* Let V be a finite-dimensional vector space and suppose $T : V \to V$ is linear. Show that (nonzero) eigenvectors of T corresponding to distinct eigenvalues are linearly independent.

5. *(15 = 7 + 8 points)* Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A^T$

(a) Calculate $A = [T]_\beta^\beta$, where β is the standard basis of $M_{2\times 2}$:

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(b) Is T diagonalizable? Why or why not?

6. *(15 points, 3 points each)* Mark each of the following statements as True or False. *Briefly* justify your answers

(a) Let $\beta = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis of V with dual basis $\beta^\star = \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}\}$. Then $\{\mathbf{f_1}, 2\mathbf{f_2}, 3\mathbf{f_3}\}$ is the dual basis of $\{\mathbf{v_1}, 2\mathbf{v_2}, 3\mathbf{v_3}\}$

(b) If A is invertible, then A is diagonalizable

(c) If $\{\mathbf{w_1}, \cdots, \mathbf{w_n}\}$ is a basis of W and $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ is subset of V then there exists $T : V \to W$ linear such that $T(\mathbf{v_i}) = \mathbf{w_i}$ for all $i = 1, \cdots, n$.

(d) The function $T : M_{2\times 2} \to M_{2\times 2}$ defined by $T(A) = \det(A)$ is a linear transformation

(e) If you row-reduce a matrix A to get a matrix B, then $rank(A) = rank(B)$.

7. *(15 = 9 + 3 + 3 points)* Let $V = P_n(\mathbb{R})$ and $c_0, c_1, \cdots, c_n$ be <u>distinct</u> real numbers.

   (a) For $i = 0, \cdots, n$, define $\mathbf{f_i} \in V^\star$ by $\mathbf{f_i}(p) = p(c_i)$.

   Show that $\gamma = \{\mathbf{f_0}, \cdots, \mathbf{f_n}\}$ is a basis of $V^\star$.

   Hint: For each $i = 0, \cdots, n$, apply your equation to $p = (x - c_0) \cdots (x - c_{i-1})(x - c_{i+1}) \cdots (x - c_n)$.

   (b) Deduce from (a) that there is a basis of $\beta = \{p_0, \cdots, p_n\}$ of V such that for all $i, j = 0, \cdots, n$

$$p_i(c_j) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases}$$

   (c) Conclude from (b) that for any polynomial $p \in V$, we have

$$p(x) = \sum_{i=0}^{n} p(c_i) p_i(x)$$