LECTURE 27 - EIGENVALUES AND EIGENVECTORS (SECTION 5.1)

I - EIGENVECTORS OF A LT

SO FAR IN OUR EIGENVECTOR ADVENTURE, WE WERE ABLE TO FIND THE EIGENVECTORS OF A MATRIX, BUT NOW LET'S SEE HOW TO FIND THE EIGENVECTORS OF A LT. LUCKILY WE NEVER HAVE TO DO THIS DIRECTLY.

NEAT FACT LET $T: V \to V$, β = ANY BASIS OF V, AND $A = [T]_\beta^\beta$

THEN: v IS AN EIGENVECTOR OF T ($Tv = \lambda v$) $\Leftrightarrow$ $\underbrace{[v]_\beta}_{=x}$ IS AN EIGENVECTOR OF A ($Ax = \lambda x$)

WHY?

$$Tv = \lambda v \Leftrightarrow [Tv]_\beta = [\lambda v]_\beta$$

$$\Leftrightarrow [T]_\beta^\beta [v]_\beta = \lambda [v]_\beta$$

$$\Leftrightarrow Ax = \lambda x$$

EX $T: P_2 \to P_2$, $T(p) = p + (x+1)p'$

FIND ALL THE EIGENVECTORS OF T

LET $\beta = \{1, x, x^2\}$ (STANDARD BASIS OF P_2)

$$A = [T]_\beta^\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\lambda = 1, 2, 3$$

LAST TIME

EIGENVECTORS **EX** $\lambda = 2$ $\quad NUL(A - 2I) = NUL\begin{bmatrix} 1-2 & 1 & 0 \\ 0 & 2-2 & 2 \\ 0 & 0 & 3-2 \end{bmatrix}$

$$= NUL\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \cdots = SPAN\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

BUT $\begin{bmatrix}1\\1\\0\end{bmatrix} = [p]_\beta$ FOR $p = 1+x$ (" $\begin{bmatrix}1\\1\\0\end{bmatrix} \rightsquigarrow 1+x$ ")

SO $p = 1+x$ IS AN EIGENVECTOR OF T WITH $\lambda = 2$

ANS
$$\lambda = 1 \rightsquigarrow p = 1$$
$$\lambda = 2 \rightsquigarrow p = 1+x$$
$$\lambda = 3 \rightsquigarrow p = 1+2x+x^2$$

NOTE $\{1, 1+x, 1+2x+x^2\}$ IS A BASIS OF P_2 OF EIGENVECTORS OF T.

II — DIAGONALIZATION

(WHY IS IT SO IMPORTANT THAT WE HAVE A BASIS OF EIGENVECTORS? B/C IN THAT CASE, THE MATRIX OF T BECOMES VERY NICE)

DEF T IS DIAGONALIZABLE $\Leftrightarrow$ IF THERE IS A BASIS β OF V SUCH THAT $[T]_\beta^\beta$ IS DIAGONAL

$\nearrow$ EIGENVECTORS

EX IF T IS AS ABOVE BUT THIS TIME $\beta = \{1, 1+x, 1+2x+x^2\}$

THEN CAN CHECK $[T]_\beta^\beta = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix} \leftarrow$ DIAGONAL! (EIGENVALUES)

MAIN THEOREM T IS DIAGONALIZABLE $\Leftrightarrow$ THERE IS A BASIS β OF V CONSISTING OF EIGENVECTORS OF T

WHY? ($\Leftarrow$) LET $\beta = \{v_1, \dots, v_n\}$ BE SUCH A BASIS

THEN $T(v_j) = \lambda_j v_j$ FOR SOME λ_j $(j = 1, \dots, n)$

CALCULATE
$$[T]_\beta^\beta = \begin{array}{c} \\ v_1 \\ \\ v_j \\ \\ v_n \\ \\ \end{array}\begin{bmatrix} & & 0 & & \\ & & 0 & & \\ \longrightarrow & & \lambda_j & & \\ & & 0 & & \\ & & \uparrow & & \end{bmatrix}$$
$$T(v_1) \cdots T(v_j) \cdots T(v_n)$$

BUT $T(v_j) = \lambda_j v_j = 0v_1 + 0v_2 + \cdots \underline{1} v_j + \cdots + 0v_n$

SO $$[T]_\beta^\beta = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & \lambda_N \end{bmatrix}$$ → DIAGONAL!

($\Rightarrow$) CONVERSELY, IF $\beta = \{v_1, \dots, v_N\}$ IS A BASIS FOR WHICH $[T]_\beta^\beta$ IS DIAGONAL, THAT IS:

$$[T]_\beta^\beta = \begin{array}{c|c} & \begin{array}{ccc} T(v_1) & T(v_j) & T(v_N) \end{array} \\ \begin{array}{c} v_1 \\ v_j \\ v_N \end{array} & \begin{bmatrix} d_1 & & \\ & d_j & \\ & & d_N \end{bmatrix} \end{array}$$

THEN $T(v_j) = 0v_1 + \dots + d_j v_j + \dots + 0v_N = d_j v_j$

SO v_j IS AN EIGENVECTOR OF T CORRESP TO $\lambda_j = d_j$

SO $\beta = \{v_1, \dots, v_N\}$ IS A BASIS OF EIGENVECTORS OF T ∎

(SECTION 5.2: WHEN CAN WE GUARANTEE SUCH A BASIS EXISTS?)

III — DIAGONALIZATION OF MATRICES

(EVERYTHING WE SAID ABOUT L_T IS OF COURSE TRUE FOR MATRICES)

DEF A IS DIAGONALIZABLE $\Leftrightarrow$ L_A IS DIAGONALIZABLE

MAIN THEOREM BECOMES:

FACT A IS DIAGONALIZABLE $\Leftrightarrow$ THERE IS A BASIS β OF $\mathbb{F}^N$ CONSISTING OF EIGENVECTORS OF A

EX $A = \begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}$

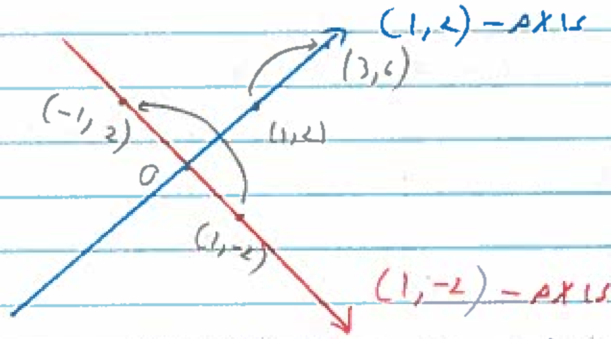
FIND A BASIS β OF $\mathbb{R}^2$ FOR WHICH $[L_A]_\beta^\beta$ IS DIAGONAL

EIGENVALUES: $\lambda = -1, 3$

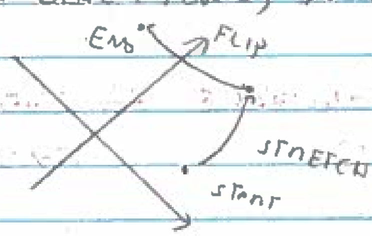
EIGENVECTORS: $\lambda = -1 \rightsquigarrow \begin{bmatrix}1\\-2\end{bmatrix}$, $\lambda = 3 \rightsquigarrow \begin{bmatrix}1\\2\end{bmatrix}$

ANS $\beta = \left\{ \begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\}$, $[L_A]_\beta^\beta = \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix} = D$

NOTE THIS GIVES US A COMPLETE INSIGHT INTO WHAT $L_A(x,y) = (x+y, 4x+y)$ DOES GEOMETRICALLY (AND EXPLAINS WHY THIS IS SO IMPORTANT)



NAMELY
- $\mathbb{R}^2$ HAS 2 AXES, ONE SPANNED BY $(1,-2)$ AND THE OTHER SPANNED BY $(1,2)$
- ON $(1,-2)$ AXIS, A FLIPS VECTORS, $AV = -V$
- ON $(1,2)$ AXIS, A STRETCHES VECTORS, $AV = 3V$
- ON OTHER POINTS, A DOES A COMBO OF THE TWO



(AMAZING, B/C BEFORE WE HAD NO IDEA WHAT A DOES)

IV - SIMILARITY

(IN FACT, IN THE CASE OF MATRICES, WE CAN SAY EVEN MORE)

FACT A DIAGONALIZABLE $\Leftrightarrow$ $A = PDP^{-1}$ FOR SOME D DIAG. P INV.

(A IS DIAGONALIZABLE IFF A IS SIMILAR TO A DIAGONAL MATRIX)

WHY? $(\Rightarrow)$

(SECTION 2.5) IF $T: V \to V$ AND β, γ ARE TWO BASES OF V,

THEN: $(*)$ $$[T]_\gamma^\gamma = P\,[T]_\beta^\beta\,P^{-1}, \quad P = {}_\gamma P_\beta$$

APPLY $(*)$ WITH: $T = L_A$

β = BASIS OF EIGENVECTORS OF A

γ = STANDARD BASIS OF $\mathbb{F}^n$

$$\Rightarrow [L_A]_\gamma^\gamma = P\,[L_A]_\beta^\beta\,P^{-1}$$

γ = STANDARD BASIS

$$A = PDP^{-1} \qquad D = [L_A]_\beta^\beta = \text{DIAGONAL}$$

(BY ASSUMPTION)

$(\Leftarrow)$ SUPPOSE $A = PDP^{-1}$, $P = [v_1 \cdots v_n]$

THEN $AP = PD$ $\qquad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$

$$\Rightarrow A[v_1 \cdots v_n] = [v_1 \cdots v_n]\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

$$\Rightarrow [Av_1 \cdots Av_n] = [d_1 v_1 \cdots d_n v_n]$$

SO $Av_j = d_j v_j$ FOR ALL j $\quad$ (EX $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = \left[5\begin{bmatrix}1\\3\end{bmatrix} \; 6\begin{bmatrix}2\\4\end{bmatrix}\right]$

HENCE $\beta = \{V_1, \ldots, V_N\}$ IS A BASIS OF $\mathbb{R}^N$ CONSISTING OF EIGENVECTORS OF A

SINCE $C = [V_1 \ldots V_N]$ IS INV

SO A IS DIAGONALIZABLE ∎