LECTURE 27 - EIGENVALUES AND EIGENVECTORS (SECTION 5.1)

I- EIGENVECTORS OF A LT

So far in our eigenvector adventure, we were able to find the eigenvectors of a matrix, but now let's see how to find the eigenvectors of a LT. Luckily we never have to do this directly.

**New Fact** Let \( T: V \to V \), \( \beta = \text{any basis of } V \), and \( A = [T]_\beta^\beta \)

Then: \( V \) is an eigenvector of \( T \) if and only if \( [V]_\beta \) is an eigenvector of \( A \).

\( (TV = AV) \)

Why? \( TV = AV \iff [TV]_\beta = [AV]_\beta \)

\( \iff [T]_\beta [V]_\beta = A [V]_\beta \)

\( \iff A x = \lambda x \)

**Ex** \( T: \mathbb{P}_2 \to \mathbb{P}_2 , \ T(p) = p + (x+1)p' \)

Find all the eigenvectors of \( T \).

**Let** \( \beta = \{1, x, x^2\} \) (standard basis of \( \mathbb{P}_2 \))

\( A = [T]_\beta^\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \) last TPE

\( \lambda = 1, 2, 3 \)

Eigenvectors: \( \lambda = 1 \) \( \text{null} (A - \lambda I) = \text{null} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \)
\[
\begin{align*}
\text{Let } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{. For } p = 1 + x, \\
\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^T = 1 + x,
\end{align*}
\]
so \( p = 1 + x \) is an eigenvalue of \( T \) with \( \lambda = 2 \).

And:
\[
\begin{align*}
\lambda &= 1 & p &= 1 \\
\lambda &= 2 & p &= 1 + x \\
\lambda &= 1 & p &= 1 + 2x + x^2,
\end{align*}
\]

Note: \( \{ 1, 1 + x, 1 + 2x + x^2 \} \) is a basis of \( P_2 \) of eigenvalues of \( T \).

**II - Diagonalization**

Why is it so important that we have a basis of eigenvalues? Big in that case, the matrix of \( T \) becomes very nice.

**Def:** \( T \) is diagonalizable if there is a basis \( \beta \) of \( V \) such that \( [T]_{\beta}^\beta \) is diagonal.

**Ex:** If \( T \) is as above but this time \( p = \{ 1, 1 + x, 1 + 2x + x^2 \} \).

Then we can check \( [T]_{\beta}^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \) is diagonal (eigenvalues).

**Main Theorem:** \( T \) is diagonalizable \( \iff \) there is a basis \( \beta \) of \( V \) consisting of eigenvalues of \( T \).

**Why?** Let \( \beta = \{ v_1, \ldots, v_n \} \) be such a basis.

Then \( \forall j \), \( T(v_j) = \lambda_j v_j \) for some \( \lambda_j \) (\( j = 1, \ldots, n \)).

Calculate \( [T]_{\beta}^\beta = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \) of \( T(v_1) \ldots T(v_n) \).

Thus \( T(v_j) = \lambda_j v_j = 0 v_1 + 0 v_2 + \cdots + 1 v_j + \cdots + 0 v_n \).
so \( [T]_p^p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{DIAGONAL} \)

\((\Rightarrow)\) CONVINFNTY, iF \( p = \{ v_1, \ldots, v_n \} \) IS A BASIS FOR WHICH \([T]_p^p\)

IS DIAGONAL, THEN \( [T]_p^p \)

\[
[T]_p^p = \begin{bmatrix}
v_1 & d_1 \\
v_2 & d_2 \\
v_n & d_n \\
\end{bmatrix}
\]

\(T(v_1)\) \(T(v_2)\) \(T(v_n)\)

Then \( T(v_j) = \lambda_1 v_1 + \ldots + \lambda_j v_j + \ldots + \lambda_n v_n = d_j v_j \)

So \( v_j \) is an EIGENVECTOR of \( T \) corresp. To \( \lambda_j = d_j \)

So \( p = \{ v_1, \ldots, v_n \} \) IS A BASIS OF EIGENVECTORS OF \( T \)

SECTION 5.2: WHEN CAN WE GUARANTEE SUCH A BASIS EXISTS?

\[ III - \text{DIAGONALIZATION OF MATRICES} \]

(EVERYTHING WE SAID ABOUT \( T \) IS OF COURSE TRUE FOR MATRICES)

DEF \( A \) IS DIAGONALIZABLE \( \Leftrightarrow \) \( A \) IS DIAGONALIZABLE

MAJ. THEOREM: RECALL:

\( \text{Fact}: \) \( A \) IS DIAGONALIZABLE \( \Rightarrow \) THERE IS A BASIS \( p \) OF \( \mathbb{F}^n \)

CONSISTING OF EIGENVECTORS OF \( A \)
EX \\
\[ A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \]

Find a basis \( \beta \) of \( \mathbb{R}^2 \) for which \( [A]_\beta \) is diagonal.

Eigenvalues: \( \lambda = -1, 3 \)

Eigenvectors: 
\( \lambda = -1 \rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix} \), \( \lambda = 3 \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)

Thus, 
\[ [A]_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = 0 \]

NOTE: This gives us a complete insight into what \( \lambda \) (\( x, y \) = (\( x+y \), \( 4x+y \)) does geometrically (and explains why this is so important).

Namely:
1) \( \mathbb{R}^2 \) has 2 axes, one spanned by \((1, 2)\) and the other spanned by \((-1, 2)\).
2) On \((1, 2)\) axis, a flip, vectors \( AV = -V \)
3) On \((1, 2)\) axis, a stretched vectors, \( AV = 3V \)
4) On other plane, \( A \) does a composite of the two.
IV - Similarity

(IN FACT, IN THE CASE OF MATRICES, WE CAN SAY EVEN MORE)

\[ \text{Fact} \quad A \text{ diagonalizable } \implies A = PDP^{-1} \text{ for some } D \text{ diag. } \tag{\ref{inv}} \]

(A is diagonalizable iff \( A \) is similar to a diagonal matrix)

WHY? \((\implies)\)

(SECTION 2.5) IF \( T : V \to V \) and \( \beta, \gamma \) are two bases of \( V \),

THEN \((\dagger)\) \( \left[ T \right]_\gamma = P \left[ T \right]_\beta P^{-1} \), \( P = \gamma \gamma \beta \)

APPLY \((\dagger)\) WITH : \( T = L_A \)

\( P = \text{Basis of Eigenvalues of } A \)

\( \gamma = \text{Standard Basis of } \mathbb{F}^n \)

\[ \implies \left[ L_A \right]_\gamma = P \left[ L_A \right]_\beta P^{-1} \]

\( \gamma = \text{Standard Basis of } \mathbb{F}^n \)

**Ball** \( A = PDP^{-1} \) \( D = \left[ L_A \right]_\beta = \text{Diagonal} \)

\((\Leftarrow)\) **Suppose** \( A = PDP^{-1} \), \( P = [v_1 \ldots v_n] \)

THEN \( AP = PD \)

\( B = \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix} \)

\[ \implies A [v_1 \ldots v_n] = [v_1 \ldots v_n] \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix} \]

\[ \implies [Av_1 \ldots Av_n] = [d_1 v_1 \ldots d_n v_n] \]

So \( Av_j = d_j v_j \) for all \( j \)

\((EX) \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 5 \times 1 & 6 \times 2 \\ 3 \times 5 & 4 \times 6 \end{bmatrix} \)
Hence \( \{v_1, \ldots, v_n\} \) is a basis of \( \mathbb{R}^n \) consisting of eigenvectors of \( A \), so \( A \) is diagonalizable since \( E = [v_1 \ldots v_n] \) is invertible.