MOCK FINAL EXAM

Instructions: This is a mock final, designed to give you some practice for the actual final. It will be similar in format to the actual final, but might have very different questions. So please also look at the study guide and the homework for a more complete study experience.

1		10
2		15
3		10
4		10
5		10
6		10
7		15
8		20
Total		100

Date: Tuesday, June 11, 2019.

1. *(10 points)* Suppose V is finite dimensional. Show that every basis of $V^\star$ is the dual basis of some basis β in V.

2. *(15 points)* If possible, find a basis β of $V = P_2(\mathbb{R})$ for which $[T]_\beta^\beta$ is diagonal, where:

$$T(ax^2 + bx + c) = cx^2 + bx + a$$

3. *(10 points)* Let $S = \{u_1, \cdots, u_n\}$ be a finite set of vectors. Show that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in Span\{u_1, \cdots, u_k\}$ for some k with $1 \leq k < n$

4. *(10 points)* Suppose $V = P_1(\mathbb{R})$ and $\beta = \{8 + x, -14 - x\}$ and $\gamma = \{2 + x, -4 + x\}$ are two different bases of V. Suppose also $T : V \to V$ is such that $[T]_\beta^\beta = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $[T]_\gamma^\gamma$.

5. *(10 points)* Let $V = P_2(\mathbb{R})$ and $T : V \to V$ be defined by:

$$T(f(x)) = f(x) + f'(x) + f''(x)$$

Find a formula for T^{-1} (or say T^{-1} does not exist)

6. *(10 points)* Suppose V is finite-dimensional and $T : V \to V$ is such that $N(T) \neq \{0\}$. Show that there is a basis β of V such that:

$$[T]_{\beta}^{\beta} = \begin{bmatrix} O & B \end{bmatrix}$$

where O is the zero matrix of size $n \times k$, where $n = \dim(V)$ and $k = \dim(N(T))$.

7. *(15 points, 3 points each)* Label each statement as True or False, and *briefly* justify your answer:

(a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

(b) There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ of rank 4

(c) If $T : V \to W$ is linear, then $T^{TT} = T$, where T^{TT} is defined as the transpose of T^T

(d) If λ is an eigenvalue of A, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$ (here λ and μ are scalars)

(e) If A is row-equivalent to B, then $Col(A) = Col(B)$

8. *(20 = 10 + 5 + 5 points)*

(a) If x_0, x_1, x_2, x_3 are given, calculate $\det(A)$, where

$$A = \begin{bmatrix} 1 & x_0 & (x_0)^2 & (x_0)^3 \\ 1 & x_1 & (x_1)^2 & (x_1)^3 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 \end{bmatrix}$$

Note: The formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ might be useful here

(b) Deduce that if x_0, x_1, x_2, x_3 are distinct, then A is invertible

(c) Conclude that if $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$ are points in $\mathbb{R}^2$ such that all the x_i are distinct (the y_i might not be), then there is a polynomial $p(x)$ of degree 3 such that $p(x_i) = y_i$ (that is, p goes through all those points)