SOLUTIONS

1. *(10 points)* Suppose V is finite dimensional. Show that every basis of V^* is the dual basis of some basis β in V.

1) LET $\gamma = \{f_1, \dots, f_N\}$ BE A BASIS OF V^*,

WHERE $N = \dim(V^*) = \dim(V)$ (SINCE $\dim(V) < \infty$)

2) LET $\gamma^* = \{F_1, \dots, F_N\}$ BE THE DUAL BASIS OF γ,

WHICH IS A BASIS OF V^{**}

3) FROM LECTURE, WE KNOW $\psi : V \to V^{**}$ IS AN ISOMORPHISM,

WHERE $\psi(x) = \hat{x} : V^* \to \mathbb{F}$, WHERE $\hat{x}(f) = f(x)$ (IF $f \in V^*$)

4) IN PARTICULAR, SINCE ψ IS ONTO AND $F_i \in V^{**}$,

$F_i = \psi(v_i) = \hat{v_i}$ FOR SOME $v_i \in V$

LET $\beta = \{v_1, \dots, v_N\}$ (WHICH IS A BASIS OF V SINCE ψ IS AN ISOMORPHISM)

5) <u>CLAIM</u> $\gamma = \beta^*$, SO γ IS THE DUAL BASIS OF β

<u>WHY?</u> WE NEED TO SHOW $f_i(v_j) = \begin{cases} 1 & \text{IF } j = i \\ 0 & \text{IF } j \neq i \end{cases}$

BUT SINCE $\{F_1, \dots, F_N\}$ IS THE DUAL BASIS OF $\{f_1, \dots, f_N\}$,

WE KNOW $F_j(f_i) = \begin{cases} 1 & \text{IF } i = j \\ 0 & \text{IF } i \neq j \end{cases}$

BUT $F_j(f_i) = \hat{v_j}(f_i) = f_i(v_j)$.

SO $f_i(v_j) = \begin{cases} 1 & \text{IF } i = j \\ 0 & \text{IF } i \neq j \end{cases} = \begin{cases} 1 & \text{IF } j = i \\ 0 & \text{IF } j \neq i \end{cases}$ ✓ ■

5) SO A BASIS OF V FOR WHICH $[T]_\beta^\beta$ IS DIAGONAL IS :

$$\boxed{\beta = \{1+x^2, x, 1-x^2\}}$$

(BECAUSE $\begin{bmatrix}1\\0\\1\end{bmatrix} = [1+x^2]_\gamma$, $\begin{bmatrix}0\\1\\0\end{bmatrix} = [x]_\gamma$, $\begin{bmatrix}1\\0\\-1\end{bmatrix} = [1-x^2]_\gamma$

2. *(15 points)* **If possible, find a basis β of $V = P_2(\mathbb{R})$ for which $[T]_\beta^\beta$ is diagonal, where:**

$$T(ax^2 + bx + c) = cx^2 + bx + a$$

1) LET $\gamma = \{1, x, x^2\}$ BE THE STANDARD BASIS OF V

2) CALCULATE $[T]_\gamma^\gamma$:

$T(1) = T(0x^2+0x+1) = x^2 = (0)(1) + (0)(x) + (1)(x^2)$

$T(x) = T(0x^2+1x+0) = x = (0)(1) + (1)x + 0(x^2)$

$T(x^2) = T(1x^2+0x+0) = 1 = (1)(1) + 0x + 0x^2$

SO $A = [T]_\gamma^\gamma = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$

3) EIGENVALUES $\quad \det(A - tI) = \begin{vmatrix}-t&0&1\\0&1-t&0\\1&0&-t\end{vmatrix} = (1-t)\begin{vmatrix}-t&1\\1&-t\end{vmatrix}$

$= (1-t)(t^2-1) = -(t-1)(t-1)(t+1) = -(t-1)^2(t+1) = 0$

$\Rightarrow \lambda = 1$ (MULT 2), $\lambda = -1$ (MULT 1)

4) EIGENVECTORS

$\lambda = 1$ $\quad E_1 = \mathrm{NUL}(A-I) = \mathrm{NUL}\begin{bmatrix}-1&0&1\\0&0&0\\1&0&-1\end{bmatrix} = \mathrm{NUL}\begin{bmatrix}1&0&-1\\0&0&0\\0&0&0\end{bmatrix}$

$x_1 - x_3 = 0 \Rightarrow x_3 = x_1 \Rightarrow \underline{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_1\end{bmatrix} = x_1\begin{bmatrix}1\\0\\1\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$

SO A BASIS FOR E_1 IS $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$\lambda = -1$ $\quad E_{-1} = \mathrm{NUL}(A+I) = \mathrm{NUL}\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix} = \mathrm{NUL}\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix} = \mathrm{SPAN}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

SO A BASIS FOR E_{-1} IS $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ (SEE TOP)

3. *(10 points)* Let $S = \{u_1, \cdots, u_n\}$ be a finite set of vectors. Show that S is linearly independent if and only if $u_1 = 0$ or $u_{k+1} \in$ *Span* $\{u_1, \cdots, u_k\}$ for some k with $1 \leq k < n$

($\Leftarrow$) IF $U_1 = 0$, THEN $S = \{U_1, \dots, U_N\} = \{0, \dots, U_N\}$ IS LD

IF $U_{K+1} \in$ SPAN$\{U_1, \dots, U_K\}$ FOR SOME K, THEN

S IS ALSO LD BECAUSE SOME VECTOR IN S,

NAMELY U_{K+1} IS A LINEAR COMBO OF THE OTHER ONES

($\Rightarrow$) BY CONTRAPOSITION, SUPPOSE $U_1 \neq 0$ AND $U_{K+1} \notin$ SPAN$\{U_1, \dots, U_K\}$

FOR ALL K WITH $1 \leq K < N$

SINCE $U_1 \neq 0$, $\{U_1\}$ IS LI

SINCE $U_2 \notin$ SPAN$\{U_1\}$ (BY ASSUMPTION WITH $K=1$),

BY THE INTRUDER THEOREM, $\{U_1, U_2\}$ IS LI

SINCE $U_3 \notin$ SPAN$\{U_1, U_2\}$ (ASSUMPTION WITH $K=2$),

BY THE INTRUDER THEOREM, $\{U_1, U_2, U_3\}$ IS LI

CONTINUING THE SAME WAY, WE EVENTUALLY GET

$S = \{U_1, U_2, \dots, U_N\}$ IS LI $\blacksquare$

3) FINALLY, $[T]_\gamma^\gamma = \mathcal{Q}\,[T]_\beta^\beta\,\mathcal{Q}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}^{-1}$

$= \begin{bmatrix} -7 & -8 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -22 & -37 \\ 16 & 27 \end{bmatrix}$

$8+x \quad -14-x$

4. *(10 points)* Suppose $V = P_1(\mathbb{R})$ and $\beta = \{$[redacted], [redacted]$\}$ and $\gamma = \{2+x, -4+x\}$ are two different bases of V. Suppose also $T : V \to V$ is such that $[T]_\beta^\beta = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $[T]_\gamma^\gamma$.

1) FIRST FIND $\mathcal{Q} = {}_\gamma\mathcal{Q}_{\leftarrow\beta}$

BY DEFINITION, $\mathcal{Q} = \big[\, [8+x]_\gamma \quad [-14-x]_\gamma \,\big]$

2) NOW FIND a & b WITH $8+x = a(2+x) + b(-4+x)$

$= 2a - 4b + x(a+b)$

$\Rightarrow \begin{cases} a+b = 1 \\ 2a-4b = 8 \end{cases}$, BUT $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -4 & 8 \end{array}\right] \xrightarrow{(\times -2)} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -6 & 6 \end{array}\right]$

$\longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{(\times -1)} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$

SO $[8+x]_\gamma = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

AND FIND c & d WITH $-14-x = c(2+x) + d(-4+x)$

$= (2c-4d) + x(c+d)$

$\Rightarrow \begin{cases} c+d = -1 \\ 2c-4d = -14 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & -4 & -14 \end{array}\right] \xrightarrow{(\times -2)} \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -6 & -12 \end{array}\right]$

$\longrightarrow \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{(\times -1)} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{cases} c = -3 \\ d = 2 \end{cases}$

SO $[-14-x]_\gamma = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ SO $\mathcal{Q} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ (SEE TOP)

THAT IS: $T^{-1}(f) = (a_0 - a_1 - 2a_2) + (a_1 - 2a_2)x + a_2 x^2$

SO $\boxed{T^{-1}(a_0 + a_1 x + a_2 x^2) = (a_0 - a_1 - 2a_2) + (a_1 - 2a_2)x + a_2 x^2}$

5. *(10 points)* Let $V = P_2(\mathbb{R})$ and $T : V \to V$ be defined by:

$$T(f(x)) = f(x) + f'(x) + f''(x)$$

Find a formula for T^{-1} (or say T^{-1} does not exist)

1) LET $\beta = \{1, x, x^2\}$ BE THE STANDARD BASIS OF V

CALCULATE $[T]_\beta^\beta$:

$T(1) = 1 + 0 + 0 = 1 = (1)(1) + 0(x) + 0(x^2)$

$T(x) = x + 1 + 0 = x + 1 = (1)1 + (1)x + (0)x^2$

$T(x^2) = x^2 + 2x + 2 = (2)(1) + (2)x + (1)x^2$

SO $A = [T]_\beta^\beta = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

2) $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $(x-2)$ $(x-2)$

$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $(x-1)$ $\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

HENCE $A^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = [T^{-1}]_\beta^\beta$

3) NOW SUPPOSE $f = a_0 + a_1 x + a_2 x^2$, THEN $[f]_\beta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ (SEE TOP)

SO $[T^{-1}(f)]_\beta = [T^{-1}]_\beta^\beta [f]_\beta = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 - a_1 - 2a_2 \\ a_1 - 2a_2 \\ a_2 \end{bmatrix}$

6. *(10 points)* Suppose V is finite-dimensional and $T : V \to V$ is such that $N(T) \neq \{0\}$. Show that there is a basis β of V such that:

$$[T]_\beta^\beta = [O \quad B]$$

where O is the zero matrix of size $n \times k$, where $n = \dim(V)$ and $k = \dim(N(T))$.

LET $\{v_1, \dots, v_k\}$ BE A BASIS OF $N(T)$ $\quad (k = \mathrm{DIM}(N(T)))$

IN PARTICULAR, $T(v_j) = 0$ FOR $j = 1, \dots, k$

EXTEND THIS TO A BASIS $\{v_1, \dots, v_n\}$ OF V $\quad (n = \mathrm{DIM}(V))$

CLAIM $\beta = \{v_1, \dots, v_n\}$ IS OUR DESIRED BASIS

WHY? WE KNOW $T(v_j) = 0$ FOR $j = 1, \dots, k$

WHICH MEANS $T(v_j) = 0 v_1 + \dots + 0 v_n$ FOR $j = 1, \dots, k$

AND FOR $j = k+1, \dots, n$, $T(v_j) = * v_1 + \dots + * v_n$ FOR SOME CONSTANTS $*$

HENCE

$$[T]_\beta^\beta = \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} \left[\underbrace{\begin{matrix} 0 & \cdots & 0 \\ 0 & O & \vdots \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix}}_{= O} \quad \underbrace{\begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix}}_{= B} \right]$$

$$\begin{array}{cc} T(v_1) \cdots T(v_k) & T(v_{k+1}) \cdots T(v_n) \end{array}$$

$$= [O \quad B]$$, WHERE O IS INDEED OF SIZE $n \times k$

7. *(15 points, 3 points each)* Label each statement as True or False, and *briefly* justify your answer:

FALSE

(a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

$\mathbb{R}^2$ ISN'T EVEN A SUBSET OF $\mathbb{R}^3$!

$(1,2) \neq (1,2,3)$

FALSE

(b) There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ of rank 4

BY RANK-NULLITY, $\dim(N(T)) + \text{rank}(T) = \dim(V)$

$\dim(N(T)) + 4 = 3$

$\Rightarrow \dim(N(T)) = -1 \Rightarrow\Leftarrow$

FALSE

(c) If $T : V \to W$ is linear, then $T^{TT} = T$, where T^{TT} is defined as the transpose of T^T

$T : V \to W$, SO $T^T : W^* \to V^*$, SO

$T^{TT} : (V^*)^* \to (W^*)^*$, THAT IS $T^{TT} : V^{**} \to W^{**}$

SO THE DOMAIN OF T^{TT} ISN'T EVEN EQUAL TO THE DOMAIN OF T!

TRUE

(d) If λ is an eigenvalue of A, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$ (here λ and μ are scalars)

IF $Av = \lambda v$, THEN $(A - \mu I)v = Av - \mu I v$

$= \lambda v - \mu v = (\lambda - \mu) v$

SO $(A - \mu I) v = (\lambda - \mu) v$, SO v IS AN EIGENVECTOR OF $A - \mu I$ WITH EIGENVALUE $\lambda - \mu$

FALSE

(e) If A is row-equivalent to B, then $Col(A) = Col(B)$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{(\times -1)} B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

BUT $Col(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

BUT $Col(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\neq$

$$= (x_1-x_0)(x_2-x_0)(x_3-x_0)(x_2-x_1)(x_3-x_1)\begin{vmatrix}1 & x_2+x_1+x_0\\ 1 & x_3+x_1+x_0\end{vmatrix}$$

$$= (x_1-x_0)(x_2-x_0)(x_3-x_0)(x_2-x_1)(x_3-x_1)\left(x_3+\cancel{x_1}+\cancel{x_0}-x_2-\cancel{x_1}-\cancel{x_0}\right)$$

$$= \boxed{(x_1-x_0)(x_2-x_0)(x_3-x_0)(x_2-x_1)(x_3-x_1)(x_3-x_2)}$$

8. *(20 = 10 + 5 + 5 points)*

(a) If x_0, x_1, x_2, x_3 are given, calculate $\det(A)$, where

$$A = \begin{bmatrix}1 & x_0 & (x_0)^2 & (x_0)^3\\ 1 & x_1 & (x_1)^2 & (x_1)^3\\ 1 & x_2 & (x_2)^2 & (x_2)^3\\ 1 & x_3 & (x_3)^2 & (x_3)^3\end{bmatrix}$$

Note: The formula $a^3-b^3=(a-b)(a^2+ab+b^2)$ might be useful here

(b) Deduce that if x_0, x_1, x_2, x_3 are distinct, then A is invertible

(c) Conclude that if $(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3)$ are points in $\mathbb{R}^2$ such that all the x_i are distinct (the y_i might not be), then there is a polynomial $p(x)$ of degree 3 such that $p(x_i)=y_i$ (that is, p goes through all those points)

(a)
$$\begin{array}{c} \\ (x-1)\\ (x-1)\\ (x-1)\end{array}\begin{vmatrix}1 & x_0 & x_0^2 & x_0^3\\ 1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 1 & x_3 & x_3^2 & x_3^3\end{vmatrix} = \begin{vmatrix}1 & x_0 & x_0^2 & x_0^3\\ 0 & x_1-x_0 & x_1^2-x_0^2 & x_1^3-x_0^3\\ 0 & x_2-x_0 & x_2^2-x_0^2 & x_2^3-x_0^3\\ 0 & x_3-x_0 & x_3^2-x_0^2 & x_3^3-x_0^3\end{vmatrix}$$

$$= \begin{vmatrix}x_1-x_0 & (x_1-x_0)(x_1+x_0) & (x_1-x_0)((x_1)^2+x_1x_0+(x_0)^2)\\ x_2-x_0 & (x_2-x_0)(x_2+x_0) & (x_2-x_0)((x_2)^2+x_2x_0+(x_0)^2)\\ x_3-x_0 & (x_3-x_0)(x_3+x_0) & (x_3-x_0)((x_3)^2+x_3x_0+(x_0)^2)\end{vmatrix}$$

$$= (x_1-x_0)(x_2-x_0)(x_3-x_0)\begin{vmatrix}1 & x_1+x_0 & (x_1)^2+x_1x_0+(x_0)^2\\ 1 & x_2+x_0 & (x_2)^2+x_2x_0+(x_0)^2\\ 1 & x_3+x_0 & (x_3)^2+x_3x_0+(x_0)^2\end{vmatrix}\begin{array}{c} \\ (x-1)\\ (x-1)\end{array}$$

$$= (x_1-x_0)(x_2-x_0)(x_3-x_0)\begin{vmatrix}\cancel{1} & \cancel{x_1+x_0} & \cancel{(x_1)^2+x_1x_0+(x_0)^2}\\ 0 & x_2-x_1 & (x_2)^2-(x_1)^2+x_2x_0-x_1x_0\\ 0 & x_3-x_1 & (x_3)^2-(x_1)^2+x_3x_0-x_1x_0\end{vmatrix}$$

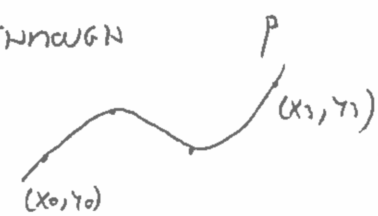
$$= (x_1-x_0)(x_2-x_0)(x_3-x_0)\begin{vmatrix}x_2-x_1 & (x_2-x_1)(x_2+x_1)+x_0(x_2-x_1)\\ x_3-x_1 & (x_3-x_1)(x_3+x_1)+x_0(x_3-x_1)\end{vmatrix}$$

(SEE TOP)

(b) $\quad \mathrm{DET}(A) = (x_1 - x_0)(x_2 - x_0)(x_3 - x_0)(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$

AND THIS IS $\neq 0$ IF x_0, x_1, x_2, x_3 ARE ALL DISTINCT

(c) SUPPOSE $p = a + bx + cx^2 + dx^3$ GOES THROUGH $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$



THEN
$$p(x_0) = a + bx_0 + c(x_0)^2 + d(x_0)^3 = y_0$$
$$p(x_1) = a + bx_1 + c(x_1)^2 + d(x_1)^3 = y_1$$
$$p(x_2) = a + bx_2 + c(x_2)^2 + d(x_2)^3 = y_2$$
$$p(x_3) = a + bx_3 + c(x_3)^2 + d(x_3)^3 = y_3$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & x_0 & (x_0)^2 & (x_0)^3 \\ 1 & x_1 & (x_1)^2 & (x_1)^3 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\underline{b}}$$

SINCE $\mathrm{DET}(A) \neq 0$ BY (b), A IS INVERTIBLE, $A\underline{x} = \underline{b}$ HAS A UNIQUE SOLUTION $\underline{x} = A^{-1}\underline{b} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, WHICH GIVES YOU THE UNIQUE POLYNOMIAL $p = a + bx + cx^2 + dx^3$ ∎