5.1.15a. By induction. Clear for $m = 1$. If $m = k+1$, $T^m(x) = T(T^k(x)) = T(\lambda^k x) = \lambda^k T(x) = \lambda^{k+1}x = \lambda^m x$.

5.1.16. (a) Let A and B be similar matrices, with $A = QBQ^{-1}$. Then $\mathrm{Tr}(A) = \mathrm{Tr}(Q^{-1}BQ) = \mathrm{Tr}(QQ^{-1}B) = \mathrm{Tr}(B)$.

(b) The trace of a linear operator should be defined as the sum of its eigenvalues (with multiplicity). From problem 12, we know that this definition is independent of similarity transforms. It is easy to see that if a linear operator T is diagonalizable, then this definition agrees with the usual definition of trace. Later in the book, we will see that this definition agrees with the usual one for all linear operators.

5.1.22a. Let $g(t) = a_0 + a_1 t + \ldots a_n t^n$. Then $g(T)(x) = (a_0 + a_1 T + \ldots a_n T^n)(x) = a_0 x + a_1 T(x) + \ldots + a_n T^n(x) = a_0 x + a_1 \lambda x + \ldots + a_n \lambda^n x = (a_0 + a_1 \lambda + \ldots + a_n \lambda^n)x = g(\lambda)x$.

5.1.23. We show that $f(T)$ and T_0 agree on a basis, which is sufficient, since both are linear operators. Let β be an eigenbasis for T, which exists since T is diagonalizable. Let v be an arbitrary element of β, with eigenvalue λ. Then $f(T)(v) = f(\lambda)(v)$, by exercise 22a, and then, since the eigenvalues of T are the roots of the characteristic polynomial, $f(\lambda) = 0$, and so $f(T)(v) = 0$. Thus, $f(T)$ is 0 on every element of β, and is thus equal to T_0.

5.2.1. (a) F, (b) F, (c) F, (d) T, (e) T, (f) F, (g) T, (h) T, (i) F.

5.2.2a. A has characteristic polynomial $(\lambda - 1)^2$. The equation $A(x,y) = (x,y)$ is equivalent to $x + 2y = x$ and $y = y$. The first equation yields $y = 0$. Thus, $(1,0)$ spans the eigenspace E_1, and so A does not have an eigenbasis, and is not diagonalizable.

5.2.2c. A has characteristic polynomial $\lambda^2 - 3\lambda - 10$, with roots -2 and 5. Thus it must have an eigenbasis. Eigenvectors are obtained by solving: $A(x,y) = -2(x,y)$, so $x + 4y = -2x$, and $y = -3x/4$, so $(4,-3)$ is an eigenvector; $A(x,y) = 5(x,y)$, so $x + 4y = 5x$, so $x = y$, and $(1,1)$ is an eigenvector. Thus, $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$, and $Q = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$.

5.2.2e. A has characteristic polynomial $1 - \lambda + \lambda^2 - \lambda^3$. By inspection, it has 1 as a root. Factoring that out, we get $1 + \lambda^2$. Since this does not split over $\mathbb{R}$, A is not diagonalizable over $\mathbb{R}$, although it is over $\mathbb{C}$.

5.2.3a. Writing $f(x)$ as $a + bx + cx^2 + dx^3$, $T(f(x)) = b + 2cx + 3dx^2 + 2c + 6dx = (b+2c) + (2c+6d)x + 3dx^2$. If a polynomial, f, has a non-zero x^3 term, then $T(f)$ has a non-zero x^2 term, but no x^3 term, so f cannot be an eigenvector. Thus, any eigenvector of T has no x^3 term, and T cannot have an eigenbasis.

5.2.3f. It is clear that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ have eigenvalue 1, and that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has eigenvalue -1. Thus, these vectors give us β.

5.2.7. We know from 2c that we can write $A = QDQ^{-1}$, where $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$, and $Q = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$. Then $A^n = (QDQ^{-1})^n = QD^nQ^{-1} = Q\begin{pmatrix} (-2)^n & 0 \\ 0 & 5^n \end{pmatrix}Q^{-1}$, which can also be written as

$$\begin{pmatrix} (-2)^n - 5^n/3 & 4(-2)^n + 5^n \\ -3(-2)^n/4 - 5^n/3 & -3(-2)^n + 5^n \end{pmatrix}.$$

5.2.12. The argument of 5.1.8 shows that every eigenvector of T with eigenvalue λ is an eigenvector of T^{-1} with eigenvalue λ^{-1}. This shows that the eigenspaces in question are equal. For part (b), let $\{v_1, \ldots, v_n\}$ be an eigenbasis for T. Then the v_i's are still eigenvectors of T^{-1}, by part (a), and still linearly independent, so they are an eigenbasis for T^{-1}.

5.2.13. (a) If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then E_1 is spanned by $(1,0)$ but E_1' is spanned by $(0,1)$.

(b) $\dim(E_\lambda') = \dim(N(A^t - \lambda I)) = \dim(N(A^t - (\lambda I)^t)) = \dim(N((A - \lambda I)^t)) = \dim(V) - \text{Rank}((A - \lambda I)^t) = \dim(V) - \text{Rank}(A - \lambda I) = \dim(N(A - \lambda I)) = \dim(E_\lambda)$.

(c) If A is diagonalizable, then its characteristic polynomial splits, and each eigenspace has full dimension. Since A^t has the same characteristic polynomial, its characteristic polynomial also splits, and since each of its eigenspaces has the same dimension as the corresponding eigenspaces for A, its eigenspaces also have full dimension, and so it splits.

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(written by Janak Ramakrishnan)

5.2.18a. We show $TU = UT$ by considering action on elements of β. As β is a basis, this is sufficient. Let v be an arbitrary element of β. Let λ be the eigenvalue of T corresponding to v, and μ the eigenvalue of U corresponding to v. Then $TUv = T(\mu v) = \mu Tv = \mu\lambda v = \lambda\mu v$. $UTv = U(\lambda v) = \lambda Uv = \lambda\mu v$.

5.2.19. Let β be an ordered basis in which $[T]_\beta$ is diagonal. By 5.1.15(a), every $v \in \beta$ is an eigenvector of T^m for any $m > 0$, showing that $[T^m]_\beta$ is also diagonal.

5.2.20. We prove the forward direction first. Let β_i be a basis of W_i, for $1 \leq i \leq k$. It is clear that the β_i's are pairwise disjoint, since $W_j \cap \sum_{i \neq j} W_i = \{0\}$, so certainly $W_j \cap W_i = \{0\}$. Let $\alpha = \beta_1 \cup \ldots \cup \beta_k$. It is clear that $|\alpha| = \sum_{i=1}^k |\beta_i| = \sum_{i=1}^k \dim(W_i)$, and by Theorem 5.10, α is a basis for V, so $\dim(V) = |\alpha| = \sum_{i=1}^k \dim(W_i)$.

For the reverse direction, take the same setup of β_i's and α. Once again, it is clear that α spans V, and since $|\alpha| \leq \sum_{i=1}^k |\beta_i| = \sum_{i=1}^k \dim(W_i) = \dim(V)$, α is a basis of V. Now by Theorem 5.10 again, $V = \bigoplus_{i=1}^k W_i$.

5.2.22. First we show that $\mathrm{Span}(\{x \in V \mid x \text{ is an eigenvector of } T\}) = \sum_{i=1}^k E_{\lambda_i}$. This is clear, since a vector is in the first space if and only if it is a linear combination of eigenvectors of T, which is precisely the set of vectors in the second space. Next we show that $E_{\lambda_j} \cap \sum_{i \neq j} E_{\lambda_i} = \{0\}$, for any j. Fix an arbitrary j, and let $v \in E_{\lambda_j} \cap \sum_{i \neq j} E_{\lambda_i}$. We will show $v = 0$. We can write $v = \sum_{i \neq j} x_i$, where each x_i is in E_{λ_i}. Then $0 = -v + \sum_{i \neq j} x_i$. If any of the vectors on the right side is nonzero, then we would have a linear combination of eigenvectors which was 0, which is impossible by Theorem 5.5. Thus, all the vectors must be 0, so $v = 0$.

5.4.1. (a) F, (b) T, (c) F, (d) F, (e) T, (f) T, (g) T.

5.4.2ace. (a) Yes – T can only decrease the degree of an element of V, and $P_2(R)$ is closed downwards with respect to degree. (c) Yes – $T(V) \subseteq W$, so $T(W) \subseteq W$. (e) No – $T\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, so $T(W) \not\subseteq W$.

5.4.4. Since W is a subspace, it is closed under linear combination. Since $g(T)(w)$, for $w \in W$, is a linear combination of vectors of the form $T^m(w)$ for $m \geq 0$, we need only show that $T^m(w) \in W$ for $m \geq 0$ and every $w \in W$. Show this by induction on m. For $m = 0$, $T^m(w) = w \in W$. For $m = k+1$, we have $T^m(w) = T^k(T(w))$. Since $T(w) \in W$, by induction we know that $T^k(T(w)) \in W$, and so we are done.

5.4.6bd. (b) $T(z) = 6x$, and $T^2(z) = 0$. Then $\{x^3, x\}$ is a basis. (d) $T(z) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, and $T^2(z) = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, so $\left\{z, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\right\}$ is a basis.