

# LECTURE 1 - INTRODUCTION

Monday, September 16, 2019 4:04 PM

## I- INTRODUCTION

Hello everyone and welcome to Math 112A, the wonderful world of PDEs! My name is Peyam and I'll be your instructor this quarter

First of all, notice the Pi in my name, and that is no coincidence because I love math ( $\pi$ ) and I love food ( $\pi$ ).

By the way, I know people love calling me Prof. Tabrizian, but please call me Peyam, or Dr. Peyam, like my awesome YouTube channel which hit 30,000 subscribers last week! (Woohoo!!!)

Alright, enough about me! Let's talk logistics! All the info you can find on the syllabus, which is on my website.

- **Course:** This course meets MWF here, but whether you show up is entirely up to you! I will post lecture notes, which you'll find online, so you won't have to come if you don't want to. Discussion sections are optional as well, except for quizzes
- **Office Hours:** W 12:30 - 1:30pm and Th 2-3 pm in 410N Rowland (rhymes with 410). Please come, I'd be happy to help you out!
- **Textbook:** The official textbook we'll be using is *Partial Differential Equations* by Walter Strauss (which is the book I used when I took this course 12 years ago!). Do **NOT** buy the Weinberger book, as we won't be using it. The textbook is very dense and confusing, so don't be shocked if you don't understand it; as long as you understand my lectures and the homework, you should be fine.
- **Grading:** How can you get that A in my course?

HW 15 %, due every Thursday during discussion, except for Midterm week, and Thanksgiving week, lowest HW dropped

Quizzes 15 %, same rules as homework. The quizzes are based on the HW, so as long as you take your homework seriously, you should be fine!

Midterm 25 %, on Wednesday, October 30 during class

Final 45 %, on Wednesday, December 11, 4-6 pm. Your final score can replace your midterm score if you do better on it!

- **Grades:** This class is going to be curved, according to the standard math department curve, which is 20 % A, 25 % B, 30 % C, 15 % D, 10 % F

## II- WHAT IS A PDE?

Previously on Math 3D, you learned about ODEs, which are equations that relate a function with one or more of its derivatives

Ex:  $y'' + 4y = 0$  (Math 3D)

Well, a PDE is just a multivariable calculus analog of an ODE

**Definition:** A **Partial Differential Equation** (PDE) is an equation relating a function  $u$  with one or more of its partial derivatives.

Example:  $u = u(x,y)$  (function of two space variables  $x$  and  $y$ )  
 $u_{xx} + u_{yy} = 0$  (Laplace's equation)

You might say: Why is this useful?

Well, let me ask you this: Are we ants living in a 1 dimensional

world, or anteaters living in a 3d world?

As my PhD advisor once said, if you can solve all PDE, then you understand the universe. This means, on the one hand, that PDE are freaking hard to study, but also that many physical phenomena are encoded in PDE.

## WHY STUDY PDE?

They arise in:

- 1) Physical Sciences (Ex: Schroedinger equation in quantum mechanics, Euler's equation, Maxwell's equation in light, Einstein's equations in relativity, Navier-Stokes to predict weather)
- 2) Geometry (Perelman solved the Poincare Conjecture, a conjecture that has been open for 200 years, using the Ricci Flow and area-minimizing surfaces)
- 3) Probability (Brownian Motion, Stochastic PDE, Airplane Simulation)
- 4) Optimization Theory (When Maximize/Minimize a certain function, a "Hamilton Jacobi equation" appears)
- 5) Image Processing (MRIs work because of PDE)
- 6) **MONEY** (Black-Scholes equation in finance)

As you can see, each PDE is its own universe, and solving a PDE amounts to explaining the universe

On the other hand, we don't understand the universe, so we probably don't understand PDEs

## EXAMPLES OF PDES:

**Ex1:**  $u = u(x,t)$  ( $x$  = Space,  $t$  = time)  
 $u_t + 3 u_x = 0$  (Transport Equation) (CHAPTER 1)

**Ex2:**  $u_t = u_{xx}$  (Heat/Diffusion Equation) (CHAPTERS 2, 3, 4)

**Ex3:**  $u_{tt} = u_{xx}$  (Wave Equation, **VERY** different from heat equation)

**Ex4:**  $u = u(x, y)$

$u_{xx} + u_{yy} = 0$  (Laplace's Equation) (CHAPTER 6)

The next ones are beyond the scope of the course:

**Ex5:**  $u_{xx} + u_{yy} = f(x, y)$  (Poisson's equation)

**Ex6:**  $u = u(x, t)$ ,  $H = H(p, x)$  ("Hamiltonian")

$u_t + H(u_x, x) = 0$  (Hamilton-Jacobi Equation)

**Ex7:**  $u_t + u u_x = 0$  (Burgers' equation, useful for traffic flow)

**Ex8:**  $u = u(x, y)$

$u_{xx} u_{yy} - (u_{xy})^2 = f(x, y)$  (Monge-Ampere Equation, **insanely** hard to solve)

**Ex9:**  $u = u(x, t)$

$u_t + u u_x + u_{xxx} = 0$  (KdV equation, notice 3  $x$ 's)

**Ex10:**  $u_t = -iu_{xx}$  (Schroedinger equation in Quantum Mechanics)

**Ex11:** The PDE that got me the PhD

$$u_t = \left( \frac{\sigma}{\tau} u_x \right)_x$$


So as you can see, the world of PDEs is an exciting world filled with lots of little adventures

**HOW TO STUDY PDES?**

- 1) Find exact answers: This is mostly what this course is about. As you'll see soon, PDEs are not as easy as ODEs, and in fact we'll have lots of trouble with even the simplest equations. Also, we'll discover Fourier series, which is kind of like an exact solution
- 2) Numerical Methods: Very popular nowadays, using the computer! Can you find an approximate solution?
- 3) Asymptotic Methods: Can I solve a more complicated PDE by perturbing (= slightly modifying) an easy PDE?
- 4) Theory of PDE: Deduce properties of solutions without even solving them (Ex: Is there a unique solution?, will do a tiny bit of this, but that's mostly left for graduate PDE classes)

### III- SOME SIMPLE PDE

I'm sure you're dying to solve some PDEs, so let's start this course by solving some very simple ones, and then I'll move on with more generalities

Ex 1:  $u = u(x, y)$  

Solve  $u_x = 0$    $u = C$

$u_x = 0 \Rightarrow u$  only depends on  $y$

$\Rightarrow u(x, y) = f(y) \quad f = \text{ANY FUNCTION}$

Ex 2:  $u_{xx} = 0 \Rightarrow u_x = f(y) \leftarrow \text{Think constant}$

$\Rightarrow u(x, y) = f(y)x + g(y)$

Ex 3:  $u_{xx} + u = 0$

(Note: Math 3D:  $y'' + y = 0 \Rightarrow y = A \cos + B \sin$ )

Here the coefficients are independent of  $x$ , so only depend on  $y$

SOLUTION

$$u(x, y) = A(y) \cos(x) + B(y) \sin(x)$$

Ex 4:  $u_{xy} = 0 \Rightarrow (u_x)_y = 0$

$\Rightarrow V_y = 0$ , WHERE  $V = u_x$

$\Rightarrow V = f(x)$

$\Rightarrow u_x = f(x)$

$\Rightarrow u(x, y) = F(x) + G(y)$

$F =$  ANTIDERIVATIVE  
OF  $f$   
 $G =$  ARBITRARY

#### IV - TYPES OF PDEs

Just like there is a classification of species, there is a classification of PDEs

Order of a PDE: Basically the highest derivative that appears

Ex 1:  $u_{xx} + 3 u_y = 0$  Second-order PDE (we go up to the second derivative)

Ex 2:  $2 u_x + 3 u_y = 0$  First-order PDE

Ex 3:  $u_{zzzyz} = 0$  Fifth-order PDE

Higher-order PDEs are harder to solve than lower-order ones.  
In this course, will focus mainly on first-order and second-order equations

### Linear vs. Nonlinear PDE:

(basically linear if it doesn't involve any squares or other fancy terms)

**Ex 1:**  $u_{xx} + u_{yyy} = 0$  (Linear)

**Ex 2:**  $(u_x)^2 + 3e^u + u_y = 0$  (Nonlinear)

Rigorous definition:

Recall: MATH 3A:

$L$  is linear if  $L(u+v) = L(u) + L(v)$  and  $L(cu) = c L(u)$

**Definition:** A **Linear PDE** is a PDE of the form

$$L(u) = f$$

Where  $L$  is linear and  $f$  doesn't depend on  $u$

**Ex:** Check that  $u_{xx} + x^2 u_{yy} = e^y$  is linear

(Note: Sure, there's an  $x^2$ , but it is still linear in  $u$ )

(Continued in Lecture 2)