LECTURE 2 - FIRST-ORDER LINEAR EQUATIONS (I)

Thursday, September 26, 2019 9:49 AM

I- TYPES OF PDE (CONTINUED)

Linear vs. Nonlinear PDE

Definition: L is linear if L(u+v) = L(u) + L(v) and L(cu) = c L(u)

Definition: A PDE is linear if it is of the form L(u) = f, where L is linear and f does not depend on u

Note: Compare this to Ax = b in Math 3A; in fact you might say that PDEs is an infinite analog of linear algebra

Example: Is the following PDE linear?

$$U_{xx} + x^2 U_{yy} = e^y$$

This is of the form L(u) = f(x,y), where L(u) = $u_{xx} + x^2 u_{yy}$ and f(x,y) = e^y

Just need to check that L is linear:

1) $$L(u+v) = (u+v)_{xx} + x^2 (u+v)_{yy}$$

$$= u_{xx} + v_{xx} + x^2 u_{yy} + x^2 v_{yy}$$

$$= \left(u_{xx} + x^2 u_{yy}\right) + \left(v_{xx} + x^2 v_{yy}\right)$$

$$= L(u) + L(v) \checkmark$$

2) $$L(cu) = (cu)_{xx} + x^2 (cu_{yy})$$

$$= c\, u_{xx} + x^2 c\, u_{yy}$$

$$= c\left(u_{xx} + x^2 u_{yy}\right)$$

$$= c\, L(u) \checkmark$$

$$= c\,(v_{xx} + x^2 v_{yy})$$
$$= c\,L(v) \checkmark$$

Ex: Is $u_{xx} + u^2 = 2$ linear ?

Of the form $L(u) = f$, with $L(u) = u_{xx} + u^2$, and can check L is not linear, so no

And of course, nonlinear PDE are way harder than linear PDE (basically, none of the methods for linear PDE can be used to solve nonlinear PDE, that's why my life is hard)

Homogeneous/Inhomogeneous PDE

If the right-hand-side is 0, then it's homogeneous; if not it's inhomogeneous

Ex 1: $u_{xx} + u_{yy} = 0$ (Homogeneous)

Ex 2: $u_{xx} + u_{yy} = 2x$ (Inhomogeneous)

Why are linear PDE so useful?

FACT 1: For linear homogeneous PDE (= of the form $L(u) = 0$), the sum of two solutions is still a solution!

Why? If $L(u) = 0$ and $L(v) = 0$, then

$L(u+v) = L(u) + L(v) = 0 + 0 = 0$

So $u + v$ is still a solution!

(Helps us build up solutions, see Chapter 5)

Fact 2: The general solution of $L(u) = f$ is of the form

$$u = u_0 + u_p$$

Where u_0 is the general solution of $L(u) = 0$
u_p is a particular solution to $L(u) = f$

Why? See Math 3A or 3D or 121A, or my YouTube video on Homogeneous and Particular solutions (on my website)

Ex: Suppose you want to solve $u_{xy} = x^2$

The general solution of $u_{xy} = 0$ is $u_0 = F(x) + G(y)$ (last time)

A particular solution of $u_{xy} = x^2$ is $u_p = 1/3\ x^3\ y$ (pure guessing)

Hence, the general solution to $u_{xy} = x^2$ is:

$$u = u_0 + u_p = F(x) + G(y) + 1/3\ x^3\ y$$

II- REVIEW: DIRECTIONAL DERIVATIVES

For the next section (hard!), we just need to do a small review of directional derivatives from Math 2D

1) **Notation**: If $u = u(x,y)$, then

$$\boxed{\nabla u = (u_x, u_y)} \quad \text{(GRADIENT VECTOR)}$$

EX IF $u(x,y) = x^2 - y^2$, THEN $\nabla u = (2x, -2y)$

2) If **v** is a vector, then the directional derivative of u in the direction of **v** is:

$$\boxed{\nabla u \cdot v}$$

v

(Note: Strictly speaking **v** has to have length 1, but this won't really matter for the Discussion below)

Ex $\quad u(x,y) = x^2 - y^2, \quad v = (2,3),$

$$\nabla u \cdot v = (2x, -2y) \cdot (2,3) = 4x - 6y$$

(Intuitively, the directional derivative measures the rate of change of u in the **v** direction)

III- THE CONSTANT COEFFICIENT CASE

Goal: Solve equations of the form $\boxed{a u_x + b u_y = 0}$

(Constant coefficient linear homogeneous PDE)

Ex: Solve $\boxed{2 u_x + 3 u_y = 0}$

CLEVER OBSERVATION!

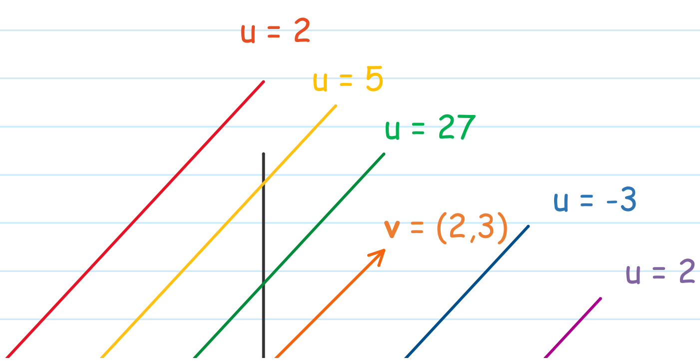
STEP 1 $\iff (u_x, u_y) \cdot (2,3) = 0$

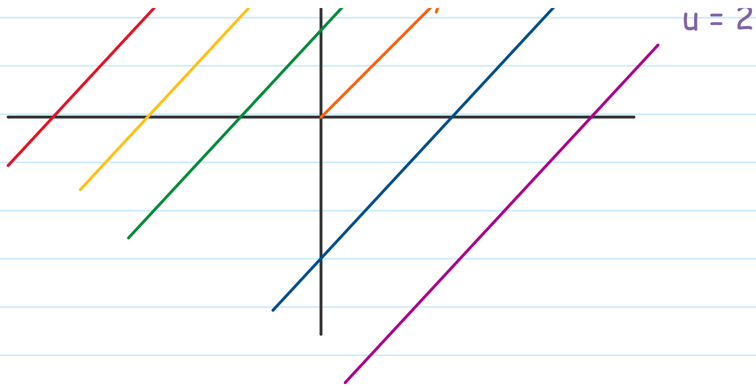
$\iff \nabla u \cdot (2,3) = 0$

POINT: Our PDE says that the directional derivative of u along **v** = (2,3) is 0

$\Longrightarrow$ u is constant along lines parallel to **v** = (2,3)

PICTURE: On each line, u is constant (the values below are for illustrative purposes only)



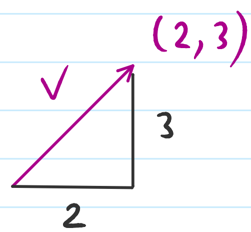


In other words, we would like to say:

$$\boxed{u(x,y) = f(?)}$$

Where ? is a variable that is constant on each line

STEP 2: How to find ?



Lines parallel to (2,3) have slope 3/2, so have equations of the form:

$$y = \frac{3}{2}x + C$$

$$\Leftrightarrow \quad 2y = 3x + 2C$$

$$\Leftrightarrow \quad 2y - 3x = \underbrace{2C}_{\text{CONSTANT}}$$

This suggests to let ? = 2y – 3x, and this is indeed what we want (a variable that is constant on those lines)

To summarize, we get our solution:

SOLUTION $\boxed{u(x,y) = f(2y-3x)}$ f = ANY FUNCTION

(and you can check that this indeed solves our PDE)

SUMMARY: The general solution of $a\,u_x + b\,u_y = 0$ is:

$$u(x,y) = f(ay-bx) \qquad (f = \text{any function})$$

Mnemonic: $$\begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx$$

Remarks:

1) Book uses f(bx−ay), which is equivalent because if f(x) is arbitrary, then so is f(−x)

2) Note that the lines $y = 3/2\, x + C$ satisfy:

$$dy/dx = 3/2 \quad (= \text{Rise/Run of } \mathbf{v} = (2,3))$$