LECTURE 3: VECTORS AND Ax = b

Monday, September 30, 2019 6:27 PM

Since we've become really good (in theory) at finding solutions to systems, we would now like to **analyze** those solutions (sort of like in bio labs where you find your specimen and then analyze it)

In other words, what does it mean geometrically for a system to have a solution?

This is why, by the way, Linear Algebra is sometimes called Linear Geometry (= study of geometric properties of solutions to equations)

I- THE MATRIX EQUATION A**x** = **b** (Section 1.4)

First of all, it turns out that there is a nicer matrix way of writing a system and its solutions:

Example: Write the following system in the form A**x** = **b** and solve it

$$\begin{cases} x + 2y - z = 1 \\ -3x - 4y + 2z = 2 \\ 5x + 2y + 3z = -3 \end{cases}$$

All this means is the following:

Let:
$$A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

"MATRIX"

"VECTORS"

Then the system is:

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}}_{\underline{b}}$$

For now, it just looks like a cosmetic fix, but we'll see in the coming weeks that there is a much deeper meaning to this.

The advantage of this is that, with this formulation, it's really easy to write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution:

$$\begin{cases} x = -4 \\ y = 4 \\ z = 3 \end{cases} \Rightarrow \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$$

VECTOR FORM

POINT: From now on, there will be no more difference between "system of equations" and "Ax = b," they literally mean the same thing.

Example: Solve $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{cccc} & & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\underline{\text{BACK SUB}}$

$$\begin{cases} x + 5z = 2 \\ y + 4z = 3 \end{cases} \Rightarrow \begin{cases} x = 2 - 5z \\ y = 3 - 4z \\ z = z \end{cases}$$

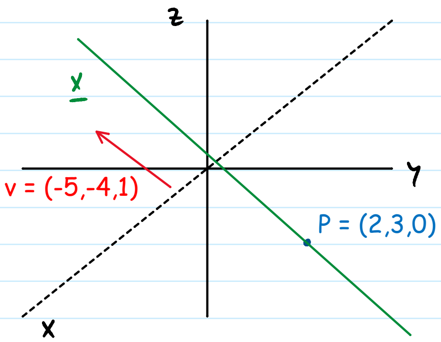
$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - 5z \\ 3 - 4z \\ 0 + z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -5z \\ -4z \\ z \end{bmatrix}$$

$$\boxed{\underline{x} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}}_{\text{Parametric vector form}}} \quad (z = \text{any number})$$

GEOMETRIC INTERPRETATION (recall introduction)

Solution is a line in R^3 (3 dimensions) going through P = (2,3,0) and parallel to **v** = (-5,-4,1)

Picture:



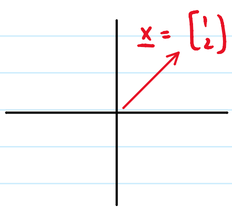
II- WHAT IS A VECTOR? (section 1.3)

Now of course you might ask: "What is a vector?"

In this course, a vector is just a list of numbers

Definition: A **vector** in R^n is a list of n numbers

Ex: $\underline{x} = \begin{bmatrix}1\\2\end{bmatrix}$ is a vector in R^2



(Note: All our vectors start at O)

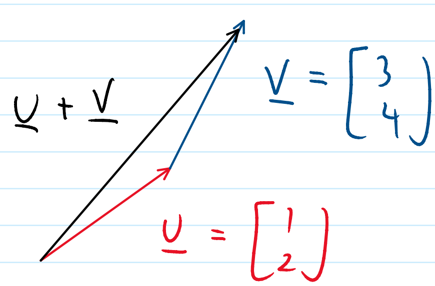
Ex: $\underline{x} = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ is a vector in R^3

Ex: $\underline{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ Zero vector

Of course, once you have vectors, you can ask:

What can you do with them?

Addition $\underset{\underline{u}}{\begin{bmatrix}1\\2\end{bmatrix}} + \underset{\underline{v}}{\begin{bmatrix}3\\4\end{bmatrix}} = \begin{bmatrix}5\\6\end{bmatrix}$



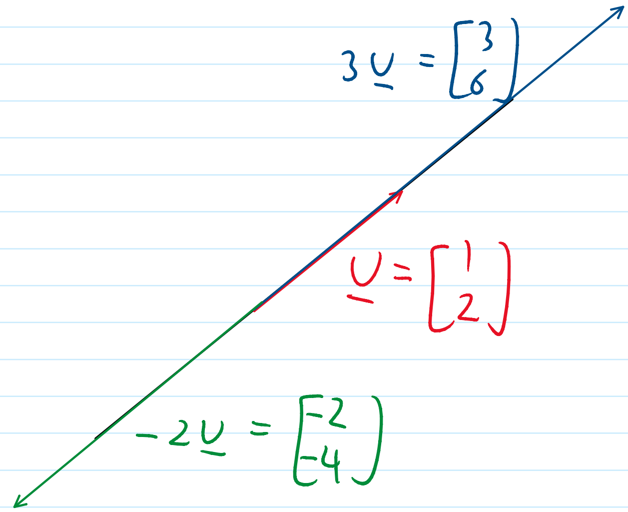
Scalar Multiplication

$$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$c \quad \underline{u}$$

$$(-2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

$$c \quad \underline{u}$$



III- LINEAR COMBINATIONS (section 1.3)

You can even combine the two operations!

Linear combination

$$5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$a \, \underline{u} + b \, \underline{v} + c \, \underline{w}$$

$\Rightarrow$

Definition: A linear combination of u, v, w is an expression of the form

$$\boxed{a\,u + b\,v + c\,w}$$

(where a, b, c are numbers)

(This definition is **REALLY** important, make sure to know it!)

Ex: $\begin{bmatrix} 6 \\ 10 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

(WITH $a = 5, \; b = -3, \; c = 2$)

IMPORTANT EXAMPLE:

Is $\begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$?

Are there numbers a, b, c such that

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + c \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix} \ ?$$

$$\Leftrightarrow \begin{cases} a - 2b - 6c = 11 \\ \quad\; 3b + 7c = -5 \\ a - 2b + 5c = 22 \end{cases}$$

SYSTEM OF EQUATIONS !!!!!

$$\left[\underbrace{\begin{matrix} 1 & -2 & 6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{matrix}}_{A} \;\middle|\; \begin{matrix} 11 \\ -5 \\ 22 \end{matrix}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & 11 \end{array}\right] \quad (*)$$

BAD WAY:

$$\xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \begin{cases} a = 9 \\ b = -4 \\ c = 1 \end{cases}$$

Answer: Yes: $9 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-4) \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$

BETTER WAY:

We don't actually care what a, b, c are, just whether they *exist* or not! That is, whether the above system is consistent or not.

Notice: In $(*)$ there are 3 pivots, so no row of the form [0 0 0 | *]

Hence the system has a solution

Hence the answer is Yes (there *exist* a, b, c such that the vector is a linear combo of the 3 other vectors)

Remarks:

1) The above observation is so important we'll isolate it as a separate fact

USEFUL TEST: If there is a pivot in every **row** of A, then Ax = b is always consistent (for any b)

2)

$$\underbrace{\begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}}_{\underline{b}}$$

is the SAME as:

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + c \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$$

So Ax = b $\Longleftrightarrow$ Is b a linear combination of the columns of A?

(Which is the desired geometric description promised at the beginning)