LECTURE 10 - THE HEAT EQUATION (II)

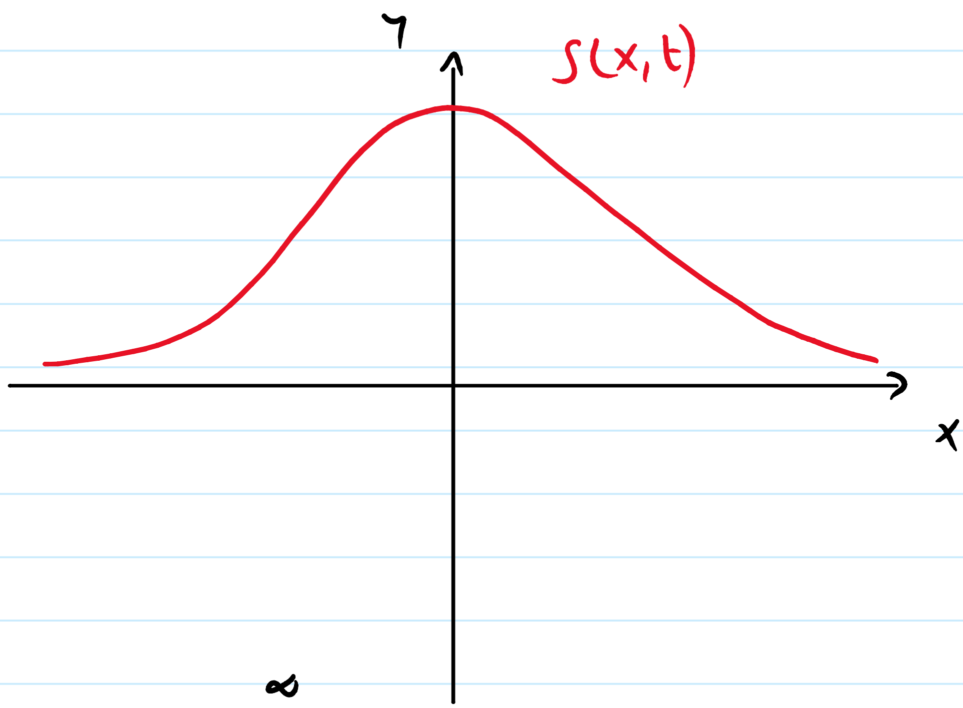
Thursday, October 17, 2019 1:10 PM

I- INITIAL-VALUE PROBLEM

Last time: Solved $u_t = k\, u_{xx}$ and found the "fundamental solution" :

$$\boxed{S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}}$$

Picture: Looks like the bell curve (here t is fixed)



Note: For all t, $\displaystyle\int_{-\infty}^{\infty} S(x,t)\, dx = 1$

(This is why we have a weird constant in front)

Goal: Solve

$$\begin{cases} u_t = k\, u_{xx} \\ u(x,0) = \phi(x) \end{cases}$$
<- Initial Condition

(So initially, the temperature is $\phi(x)$)

It turns out that we can write our solution in terms of $S(x,t)$ (which is why it's so fundamental!)

Guess: $u(x,t) = S(x,t)\, \phi(x)$

This *almost* works, provided we redefine our notion of multiplication.

In fact: $u(x,t) = S(x,t) * \phi(x)$ where $*$ is convolution!

II- CONVOLUTION

Definition: If f and g are functions, then

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

SUM = X

$$= \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

SUM = X

Note:

1) $f * g$ is a function of x (not y !)
2) The two definitions are equivalent, if you use the u-substitution u = x-y

Example:

$$f(x) = \begin{cases} e^x & \text{IF } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$g(x) = e^{3x}$

Then:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

$$= \int_{0}^{1} e^{y}\, e^{3(x-y)}\, dy$$

$(f = 0$ outside [0,1])

$$= \int_{0}^{1} e^{y}\, e^{3x}\, e^{-3y}\, dy$$

$$= e^{3x} \int_{0}^{1} e^{-2y}\, dy$$

$$= \underbrace{e^{3x}\left(-\frac{1}{2} e^{-2} + \frac{1}{2}\right)}_{\text{Function of } x}$$

Note:

1) $f * g$ measures how "similar" f and g are
2) $f * g$ is kind of like an analog of polynomial multiplication, but for functions (see last section below, or see YouTube video)

III- SOLVING OUR PDE

Now how does convolution help solve our PDE?

FACT: A solution of

$$\begin{cases} u_t = k\, u_{xx} \\ u(x,0) = \phi(x) \end{cases}$$

is: $u(x,t) = S(x,t) * \phi(x)$

$$= \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\, dy$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}}\, \phi(y)\, dy} \quad (*)$$

Note: This is basically the best we can get. We cannot get rid of the integral, except for some special situations:

IV- EXAMPLE

Example: Solve the PDE with $u(x,0) = \underbrace{e^{-x}}_{\phi(x)}$

STEP 1:

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \underbrace{e^{-y}}_{\phi(y)} \, dy$$

STEP 2: Focus on exponent

$$-\frac{(x-y)^2}{4kt} - y = -\left(\frac{(x-y)^2}{4kt} + y \right)$$

$$= -\left(\frac{\boxed{(x-y)^2 + 4kty}}{4kt} \right)$$

STEP 3: Focus on numerator

$$(x-y)^2 + 4kty = y^2 - 2xy + x^2 + 4kty$$

$$= y^2 + (4kt - 2x)y + x^2$$

(complete the square with respect to y, think x = constant)

$$= \left(y + \frac{4kt-2x}{2} \right)^2 - \left(\frac{4kt-2x}{2} \right)^2 + x^2$$

$$= (y + 2kt - x)^2 - (2kt - x)^2 + x^2$$

$$= (y+2kt-x)^2 - 4k^2t^2 + 4ktx - \cancel{x^2} + \cancel{x^2}$$

$$= (y+2kt-x)^2 + 4kt(x-kt)$$

STEP 4: Going back to STEP 2, we get:

$$-\frac{(x-y)^2}{4kt} - y = -\left(\frac{\boxed{(y+2kt-x)^2 + 4kt(x-kt)}}{4kt}\right)$$

$$= \boxed{-\frac{(y+2kt-x)^2}{4kt} - (x-kt)}$$

This was the exponent of e in our integral

STEP 5: And so, our solution becomes:

$$U(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\boxed{-\frac{(y+2kt-x)^2}{4kt} - (x-kt)}} \, dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} \, e^{kt-x} \, dy$$

(doesn't depend on y)

$$= \frac{e^{kt-x}}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\Big(\underbrace{\frac{y+2kt-x}{\sqrt{4kt}}}_{p}\Big)^2} dy$$

STEP 6: GRAND FINALE !

u-substitution:

$$p = \frac{y + 2kt - x}{\sqrt{4kt}}$$

$$dp = \frac{dy}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\, dp$$

$$p(-\infty) = -\infty, \quad p(\infty) = \infty$$

$$u(x,t) = \frac{e^{kt-x}}{\sqrt{\cancel{4}\pi \cancel{k}\cancel{t}}} \int_{-\infty}^{\infty} e^{-p^2} \cancel{\sqrt{4kt}}\, dp$$

$$= \frac{e^{kt-x}}{\cancel{\sqrt{\pi}}} \underbrace{\int_{-\infty}^{\infty} e^{-p^2} dp}_{\cancel{\sqrt{\pi}}}$$

$$\boxed{u(x,t) = e^{kt-x}}$$

TA-DAAAA!!!

(Solves $u_t = k\, u_{xx}$ with $u(x,0) = e^{-x}$)

V- WHY THIS WORKS

Why does $u(x,t) = S(x,t) * \phi(x)$ solve

$$\begin{cases} u_t = k\, u_{xx} \\ u(x,0) = \phi(x) \end{cases} \quad ?$$

Actual proof is difficult, but here's some intuition as to why it should be true

Let $u(x,t) = \displaystyle\int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\, dy$

PART 1: Check $u_t = k\, u_{xx}$

$$u_t = \left(\int_{-\infty}^{\infty} S \phi \right)_t = \int_{-\infty}^{\infty} S_t \phi$$

And

$$u_{xx} = \left(\int_{-\infty}^{\infty} S \phi \right)_{xx} = \int_{-\infty} S_{xx} \phi$$

(technically use the Chain Rule)

Hence:

$$u_t - k u_{xx} = \int_{-\infty}^{\infty} S_t \phi - k S_{xx} \phi$$

$$= \int_{-\infty}^{\infty} (\underbrace{S_t - k S_{xx}}_{0}) \phi = 0$$

(Since S solves the heat equation)

STEP 2: Check $u(x,0) = \phi(x)$

Question: What happens to $S(x,t)$ as $t \to 0^+$?

Notice the following:

1) If $x \neq 0$, then

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}} \xrightarrow{t \to 0^+} 0$$

(IF $x \neq 0$)

2) If x = 0, then

$$S(0,t) = \frac{1}{\sqrt{4\pi k t}} \xrightarrow{t \to 0^+} \infty$$

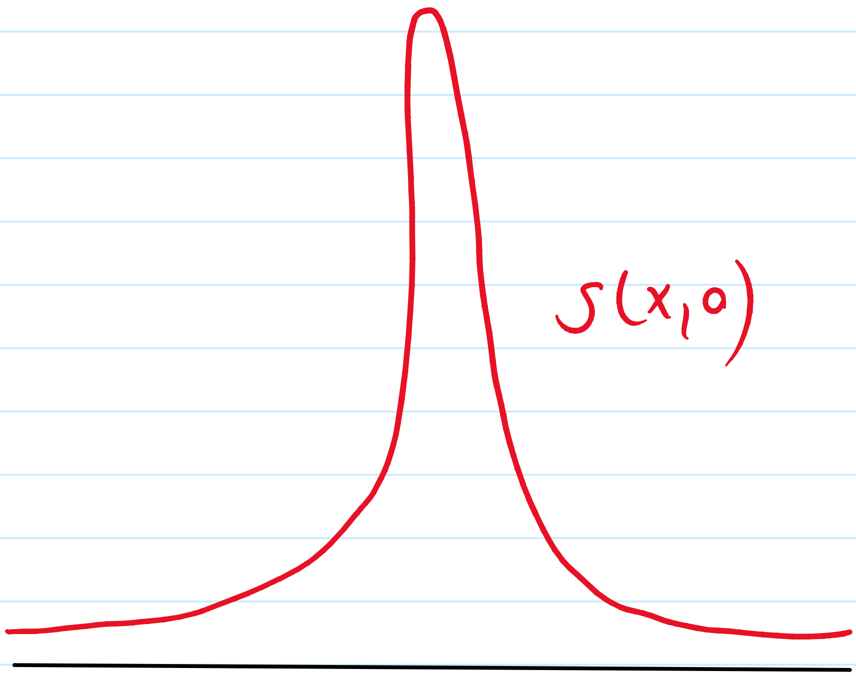
3) Finally,

$$\int_{-\infty}^{\infty} S(x,t)\, dx = 1$$

Which is true even if $t \to 0^+$

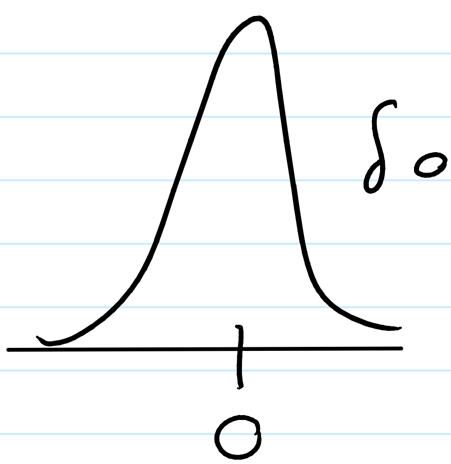
So basically S(x,0) is 0 everywhere, but has an infinite spike at x = 0, and total area = 1.

Picture:



So $S(x,0)$ - $\delta_0(x)$ where δ_0 is the Dirac Delta Functional which satisfies:

1) $\delta_0(x) = 0$ if x is not 0
2) $\delta_0(0) =$ infinity
3) δ_0 has integral 1



Therefore:

$$u(x,0) = S(x,0) * \phi(x)$$
$$= \delta_0(x) * \phi(x)$$

$$= \int_{-\infty}^{\infty} \underbrace{\delta_0(y)}_{0 \text{ except if } y = 0} \phi(x-y)\, dy$$

$$= \int_{-\infty}^{\infty} \delta_0 \, \phi(x-0)\, dy$$

$$= \int_{-\infty}^{\infty} \delta_0 \, \phi(x) \, dy$$

$$= \phi(x) \underbrace{\int_{-\infty}^{\infty} \delta_0 \, dy}_{1}$$

$$= \phi(x)$$

So indeed we get $u(x,0) = \phi(x)$

VI- CONVOLUTION INTUITION (optional)

Suppose $f(x) = a_2 x^2 + a_1 x + a_0$ and $g(x) = b_2 x^2 + b_1 x + b_0$

Find the coefficient of x^2 in $f(x)\,g(x)$

$f(x)\,g(x) = (a_2 b_0 + a_1 b_1 + a_0 b_2)\,x^2$ + other terms

So the coefficient of x^2 is:

$a_0 b_2 + a_1 b_1 + a_2 b_0$
$= a_0 b_{2-0} + a_1 b_{2-1} + a_2 b_{2-2}$

$$= \sum_{i=0}^{2} a_i \, b_{2-i}$$

In general, the coefficient of x^k in $f(x)\,g(x)$ is:

$$\sum_{i=0}^{k} a_i \, b_{k-i}$$

SUM = K

Compare to:

$$(f*g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

SUM = X

So $f * g$ is kind of like the x^{th} coefficient in the product of f and g (if you think of them as polynomials)