LECTURE 11: HEAT EQUATION PROPERTIES (I)

Friday, October 18, 2019 6:38 PM

Now that we've seen how to solve the heat equation, let's discuss some more general properties of this equation.

Note: At **NO** point are we using the fundamental solution or convolution! All our properties hold true for **any** solution of the heat equation!

In PDE, there are two main classes of methods:

1) Energy methods
2) Maximum principle methods

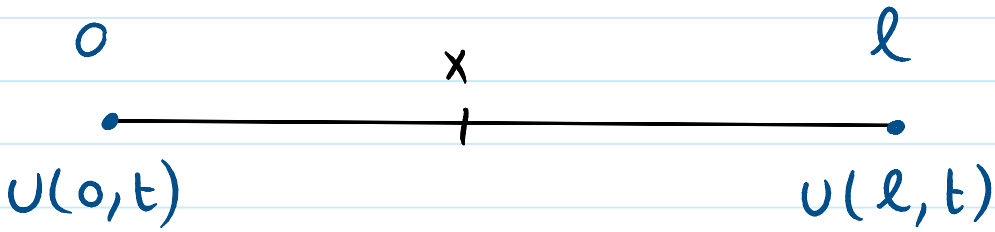
Today: Energy methods

I- ENERGY METHOD

Based on multiplying your PDE by a function and integrating by parts.

Consider a finite rod of length l with initial temperature 0 and 0 boundary conditions (= Insulated at endpoints

Picture: t fixed



Consider:

$$\begin{cases} u_t = k\, u_{xx} & (0 < x < l,\ t > 0) \\ u(x,0) = 0 & \Leftarrow \text{Initially} \\ u(0,t) = 0,\ u(l,t) = 0 & \Leftarrow \text{At endpoints} \end{cases}$$

Claim: $u(x,t) = 0$ for ALL x and t

Note: Compare with $Ax = 0 \Rightarrow x = 0$ in linear algebra
Here we're saying that $Lu = 0 \Rightarrow u = 0$, where L is our PDE with initial/boundary conditions

Why? Energy method!

STEP 1:

Start with:

$$u_t = k\, u_{xx}$$

Multiply both sides of the PDE by u:

$$u_t\, u = k\, u_{xx}\, u$$

And integrate with respect to x on $[0,l]$:

$$\underbrace{\int_0^\ell u_t u \, dx}_{A} = k \underbrace{\int_0^\ell u_{xx} u \, dx}_{B}$$

STEP 2:

Study of A:

$$\int_0^\ell U_t U \, dx = \int_0^\ell \frac{1}{2} \frac{d}{dt} (U)^2 \, dx$$

$$= \frac{d}{dt} \left[\frac{1}{2} \int_0^\ell U^2 \, dx \right]$$

STEP 3:

Study of B: Integrate by parts with respect to x to get (here boundary terms might matter)

Analogy:

$$\int_a^b f'' g = f'(b) g(b) - f'(a) g(a) - \int_a^b f' g'$$

Here:

$$\int_0^\ell u_{xx}\, u \, dx$$

$$= u_x(\ell,t)\,\cancelto{0}{u(\ell,t)} - u_x(0,t)\,\cancelto{0}{u(0,t)} - \int_0^\ell u_x\, u_x \, dx$$

(Because u(0,t) = u(l,t) = 0 by the boundary condition)

$$= -\int_0^\ell (u_x)^2 \, dx$$

STEP 4:

So (A) = k (B) says:

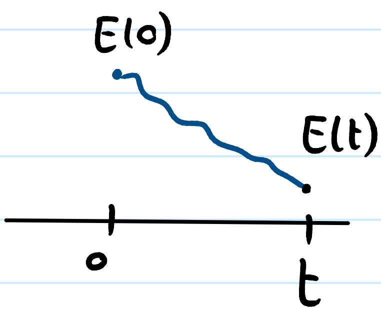
$$\frac{d}{dt}\underbrace{\left[\frac{1}{2}\int_0^\ell u^2 \, dx\right]}_{E(t)} = \underbrace{k\left(-\int_0^\ell (u_x)^2 \, dx\right)}_{\leq 0\ !}$$

Therefore: $E'(t) \leq 0$

In particular, the energy

$$E(t) = \frac{1}{2}\int_0^{\ell} \left(u(x,t)\right)^2 dx \quad \text{is decreasing!}$$

(Interpretation: Heat is dissipative. An insulated metal rod generally gets cooler with time)



So $$E(t) \leqslant E(0)$$

Therefore:

$$\boxed{\frac{1}{2}\int_0^{\ell} \left(u(x,t)\right)^2 dx \leqslant \frac{1}{2}\int_0^{\ell} \left(u(x,0)\right)^2 dx} \quad (*)$$

STEP 5:

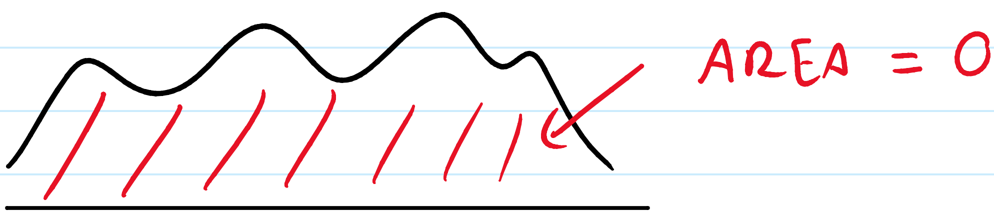
BUT u(x,0) = 0 (by the initial condition!)

Therefore:

$$0 \leq \frac{1}{2}\int_0^{\ell} \underbrace{(u(x,t))^2}_{\geq 0} dx \leq 0$$

Which implies that in fact:

$$\frac{1}{2}\int_0^{\ell} \underbrace{(u(x,t))^2}_{\geq 0} dx = 0$$



But the only way that the area under a positive function is 0 is if the function is the zero function!

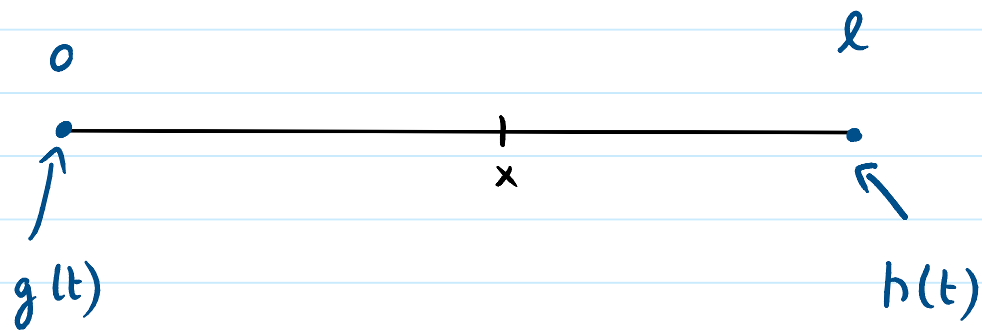
Hence $(u(x,t))^2 = 0$ for all x, so $u(x,t) = 0$ for all x (and all t)

II- UNIQUENESS

Consequence:

There is at most one solution of:

$$\begin{cases} u_t - k\, u_{xx} = f(x,t) & \text{<- Inhomogeneous (forcing) term} \\ u(x,0) = \phi(x) & \text{<- Initial Profile} \\ u(0,t) = g(t),\ u(l,t) = h(t) & \text{<- Endpoints} \end{cases}$$



Why?

Suppose u and v are two solutions, and consider $w = u - v$

Then $w_t = (u-v)_t = u_t - v_t = (k\, u_{xx} + f) - (k\, v_{xx} + f)$
$= k\, u_{xx} + f - k\, v_{xx} - f$
$= k(u-v)_{xx}$
$= k\, w_{xx}$

So w satisfies $w_t = k\, w_{xx}$

Moreover $w(x,0) = (u-v)(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) = 0$

$w(0,t) = u(0,t) - v(0,t) = g(t) - g(t) = 0$
$w(l,t) = u(l,t) - v(l,t) = h(t) - h(t) = 0$

So w satisfies:

$w_t = k\, w_{xx}$
$w(x,0) = 0$
$w(0,t) = 0,\ w(l,t) = 0$

Therefore, by the previous fact, $w(x,t) = 0$

That is, $u(x,t) - v(x,t) = 0$

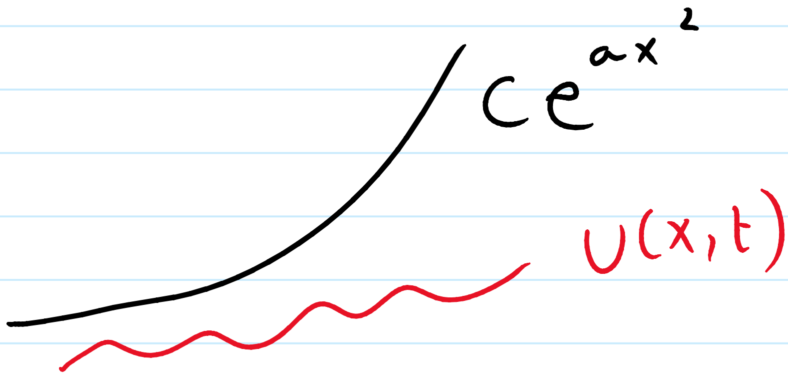
So $u(x,t) = v(x,t)$, so $u = v$

Interesting Sidenote:

For the infinite rod where -infinity < x < infinity (which we've considered last time), we **DON'T** have uniqueness, and in fact there are MANY solutions of $u_t = k\, u_{xx}$ with $u(x,0) = \phi(x)$.

So $u(x,t) = S(x,t) * F(x)$ is a solution, but there are many other ones!

BUT it turns out that there is only one solution among the ones with the property that $u(x,t) \leq C \exp(ax^2)$ for some $C > 0$ and $a > 0$



All the other solutions grow **FASTER** than $C\exp(ax^2)$, which isn't very realistic physically!

III- STABILITY

Recall the big 3 questions of PDE

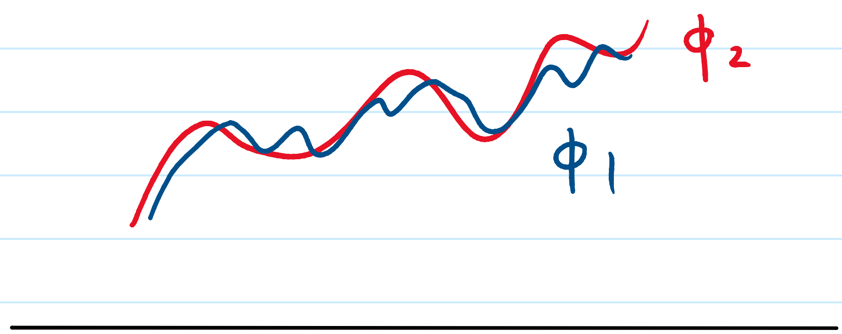
1) Existence (which we've shown for the infinite rod in 2.4, and will show for the rod of length l in Chapters 4 & 5)
2) Uniqueness (already shown)
3) Stability: If the initial conditions are close, then our solutions are close.

Suppose u and v solve the same PDE

$u_t = k\,u_{xx} + f(x,t)$
$u(0,t) = g(t),\ u(l,t) = h(t)$

But $u(x,0) = \phi_1(x)$ and $v(x,0) = \phi_2(x)$

Where ϕ_1 and ϕ_2 are "close".



Then are $u(x,t)$ and $v(x,t)$ close as well?

YES!

Why? Let $w = u - v$

Then w satisfies

$$\begin{cases} w_t = k\, w_{xx} \\ w(0,t) = 0,\ w(l,t) = 0 \\ w(x,0) = \phi_1(x) - \phi_2(x) \end{cases}$$

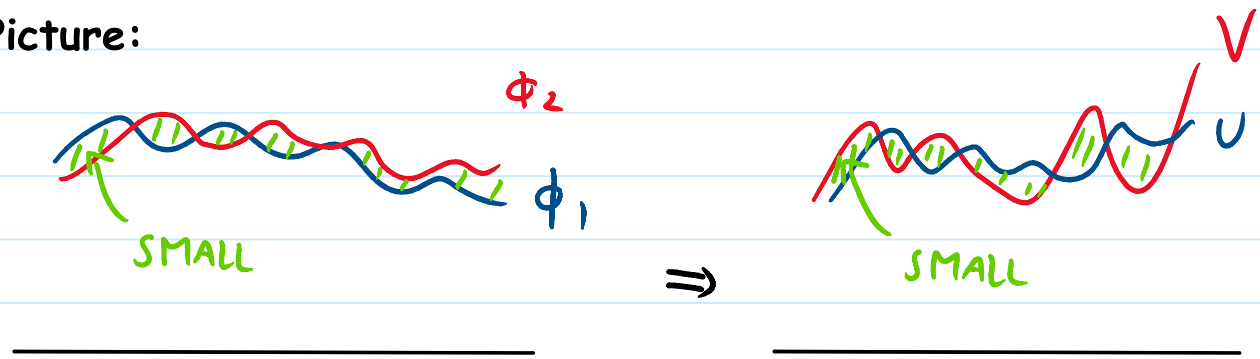
Note: In the energy method, we didn't use the initial condition until the very end.
In particular, $(*)$ is still true! (with w instead of u):

$$\cancel{\frac{1}{2}} \int_0^\ell (w(x,t))^2 \, dx \leq \cancel{\frac{1}{2}} \int_0^\ell (w(x,0))^2 \, dx$$

$$\boxed{\int_0^\ell (u(x,t) - v(x,t))^2 \, dx \leq \underbrace{\int_0^\ell (\phi_1(x) - \phi_2(x))^2 \, dx}_{\text{Only depends on } \phi_1 - \phi_2}}$$

In particular, if ϕ_1 and ϕ_2 are close, then $\phi_1 - \phi_2$ is small, therefore, by the above, u - v is small, so u and v are close!

Picture:



So indeed we have stability, but in an integral sense!

Next time: Maximum principle methods!