LECTURE 12: HEAT EQUATION PROPERTIES (II)

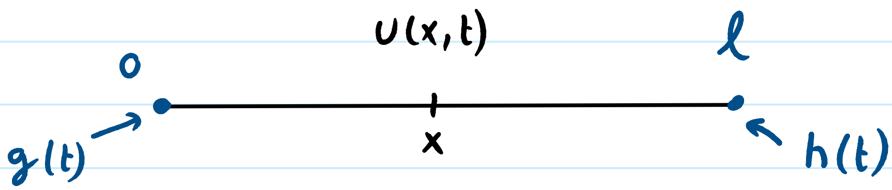
Monday, October 21, 2019 5:22 PM

Today: All about the maximum principle
(which is **VERY** different from the energy method)

I- MAXIMUM PRINCIPLE

Consider again a rod of length l, insulated at the endpoints

Picture: t fixed



Setting: Suppose u satisfies:

Terminal time

$$\begin{cases} u_t = k\, u_{xx} & (0 < x < l,\ 0 < t < T) \\ u(0,t) = g(t),\ u(l,t) = h(t) & \text{(Endpoints)} \\ u(x,0) = \phi(x) & \text{(Initially)} \end{cases}$$

Question: Where does u(x,t) attain its largest value?
(= When/where is the rod the hottest?)

Fact: [Maximum principle] (MP)

The maximum of $u(x,t)$ is attained either **initially** ($t = 0$) or at the **endpoints** ($x = 0$ or $x = l$)

That is: $\max u(x,t)$ is the **larger** one of:

$$\max g(t), \max h(t), \text{ and } \max \phi(x)$$

Picture:

The max of u on the **WHOLE** rectangle is located somewhere on the bottom or lateral sides, no need to look elsewhere!

Interpretation: A metal rod is hottest either initially, or at the endpoints (which is why you should NEVER touch a plate with your bare hands right when you take it out of the oven, or at the border!)

Example:

$$\begin{cases} u_t = k\, u_{xx} \quad (0 < x < 2,\ 0 < t < 2\pi) \\ u(0,t) = \sin(t),\ u(l,t) = 2 + \cos(t) \\ u(x,0) = 4 - x^2 \end{cases}$$

The maximum of $u(0,t) = g(t) = = \sin(t)$ is 1 (ENDPOINT)
The maximum of $u(l,t) = h(t) = 2 + \cos(t)$ is 3 (ENDPOINT)
The maximum of $u(x,0) = \phi(x) = 4-x^2$ is 4 (INITIALLY)

=> By MP, the maximum of u is the larger one of 1, 3, 4, that is 4

Remarks:

1) The same is true for min if you replace u with -u (-u also solves the heat equation), that is:

$$\boxed{\text{min } u = \text{the smaller one of min } g(t), \text{ min } h(t), \text{ min } \phi(x)}$$

2) **Sidenote:** In theory, the max could *also* be attained somewhere inside the rectangle, but we have the following result:

FACT: [STRONG Maximum Principle]

u attains its maximum **ONLY** at the bottom or the lateral sides.

In other words, if u attains its maximum inside or at the top of the rectangle, then u is constant!

II- UNIQUENESS

What's pretty amazing about this section is that we can prove the **SAME** results as last time (uniqueness, stability, etc.), but this time using the maximum principle.

Try to review this lecture and last lecture to really appreciate the similarities and differences!

Suppose u solves:

$$\begin{cases} u_t = k\, u_{xx} \\ u(0,t) = 0,\ u(l,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Claim: $u(x,t) = 0$

Why?

1) By MP, the max of u is the larger one of:

   $\max u(0,t) = \max 0 = 0$ (Endpoint)
   $\max u(l,t) = \max 0 = 0$ (Endpoint)
   $\max u(x,0) = \max 0 = 0$ (Initial)

   So max u is 0, therefore $u(x,t) \leq 0$

2) On the other hand, by the MP again, the min value of u is the smaller one of:

   $\min u(0,t) = \min 0 = 0$

$\min u(l,t) = \min 0 = 0$
$\min u(x,0) = \min 0 = 0$

Hence the min of u is 0, so $u(x,t) \geq 0$

3) Combining both, we get $u(x,t) = 0$

Consequence: Uniqueness of the heat equation (just like last time by considering $w = u - v$)

III- STABILITY

This time we still get stability, but not in an integral sense, but in a "maximal" sense.

Setting: Suppose u and v solve

$$\begin{cases} u_t - k\, u_{xx} = f(x,t) \\ u(0,t) = g(t),\ u(l,t) = h(t) \end{cases}$$

But $u(x,0) = \phi_1(x)$ and $v(x,0) = \phi_2(x)$, where ϕ_1 and ϕ_2 are "close"

Then $w = u - v$ solves:

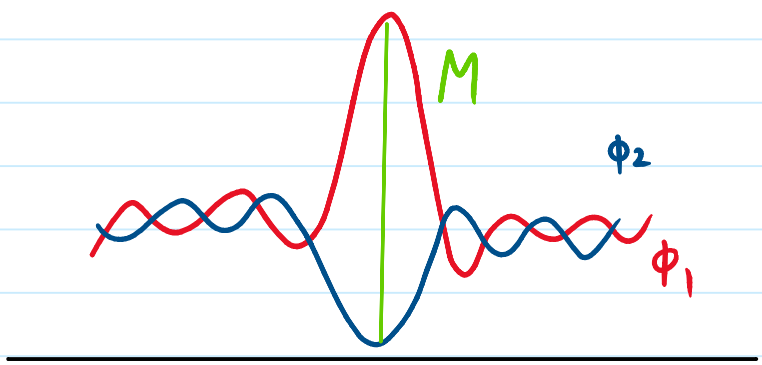
$$\begin{cases} w_t - k\, w_{xx} = 0 \\ w(0,t) = 0,\ w(l,t) = 0 \\ w(x,0) = \phi_1(x) - \phi_2(x) \end{cases}$$

Consider:

$$\boxed{M = \max |\phi_1(x) - \phi_2(x)|} \quad (\geq 0)$$

(like a "worst-case" distance/error between $\phi_1(x)$ & $\phi_2(x)$)

Picture:



1) By MP, max w is the larger of:

$$\max w(0,t) = \max 0 = 0 \leq M$$
$$\max w(l,t) = \max 0 = 0 \leq M$$
$$\max w(x,0) = \max \underbrace{\phi_1(x) - \phi_2(x)}_{w(x,0)} \leq \max |\phi_1(x) - \phi_2(x)| = M$$

(Here we used that for every z, we have $z \leq |z|$)

Hence $\max w(x,t)$ (whatever it is) is for sure $\leq M$

So $w(x,t) \leq \max w(x,t) \leq M \Rightarrow w \leq M$

2) On the other hand, by the minimum principle, min of w is the smaller one of

$\min w(0,t) = 0 \geq -M$
$\min w(l,t) = 0 \geq -M$
$\min w(x,0) = \min \underbrace{\phi_1(x) - \phi_2(x)}_{w(x,0)} \geq \min -|\phi_1(x) - \phi_2(x)| = -\max |\phi_1(x) - \phi_2(x)| = -M$

(Here we used $z \geq -|z|$ for every z, as well as $\min -z = -\max z$)

Hence $\min w(x,t) \geq -M$

So $w(x,t) \geq \min w(x,t) \geq -M \Rightarrow w \geq -M$

3) Hence $-M \leq w \leq M$, so $|w| \leq M$, which means $|u-v| \leq M$, and in particular $\max |u-v| \leq M$

4) Conclusion: For all x and t

$$\boxed{\max |u(x,t) - v(x,t)| \leq \max \underbrace{|\phi_1(x) - \phi_2(x)|}_{\text{Small}} \; (= M)}$$

Interpretation: If ϕ_1 and ϕ_2 are so close to make the worst-case error $\max |\phi_1(x) - \phi_2(x)|$ small, then the worst-case error $\max |u - v|$ is small, which means u and v are close enough as well. So here we get stability, but with a max sense

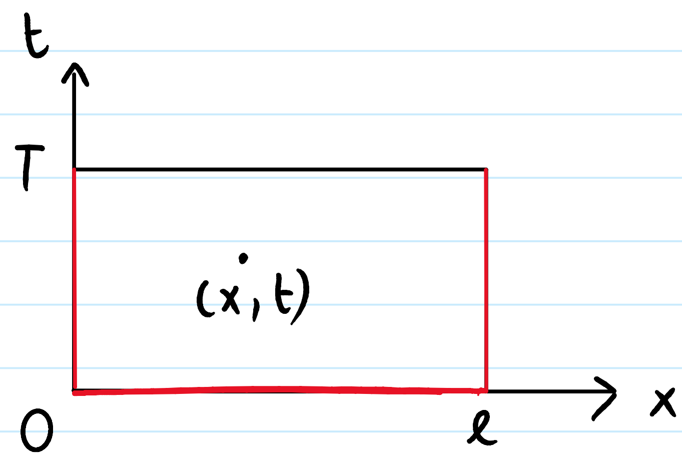
Note: Generally, use energy methods for integral results, maximum principle methods for max results.

IV- OPTIONAL: PROOF OF THE MAXIMUM PRINCIPLE

Recall: (Math 2D) If $f(x,y)$ has a maximum at (x,y), then $f_x = 0$, $f_y = 0$, and $f_{xx} \leq 0$ and $f_{yy} \leq 0$ at that point

Main idea: Suppose u has a maximum at (x,t), where (x,t) is inside the rectangle

Picture:



Then $u_t = 0$ and $u_{xx} \leq 0$ at (x,t)
Suppose for a second that we can show $u_{xx} < 0$, then we get a contradiction, because at (x,t)

$u_t - k\, u_{xx} = 0 - k\,(u_{xx}) > 0$ (by the above)

But also $u_t - k\, u_{xx} = 0$ (by the PDE!), so get $0 > 0$

This is a contradiction *unless* that maximum is attained at $x = 0$ or $x = l$ or $t = 0$ (which is what we want), or at $t = T$, the latter we have to exclude.

This ***almost*** works, except need to modify u a little bit!

Actual Proof:

STEP 1: Let $\varepsilon > 0$ be a small constant and consider:

$$\boxed{v(x,t) = u(x,t) + \varepsilon x^2}$$

STEP 2: Suppose v attains its maximum at (x,t), where (x,t) is *inside* the rectangle:

Then at (x,t), we have:

$v_t = 0$ and $v_{xx} \leq 0$, so $v_t - k\, v_{xx} = 0 - k\, v_{xx} \geq 0 \quad (*)$

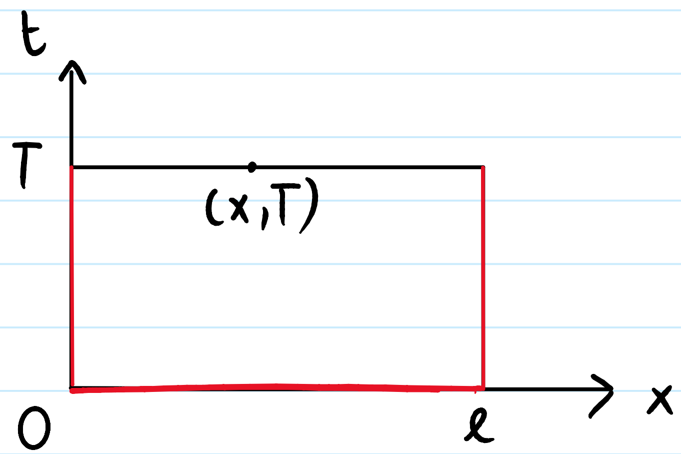
But $v_t = u_t + 0$ and $v_{xx} = u_{xx} + 2\varepsilon$

So $v_t - k\, v_{xx} = u_t - k\, u_{xx} - 2k\varepsilon = -2k\varepsilon < 0$, so we get a contradiction with $(*)$

So v must attain its maximum either initially ($t = 0$), at the endpoints ($x = 0$ or $x = l$) or terminally ($t = T$)

STEP 3: Exclude $t = T$

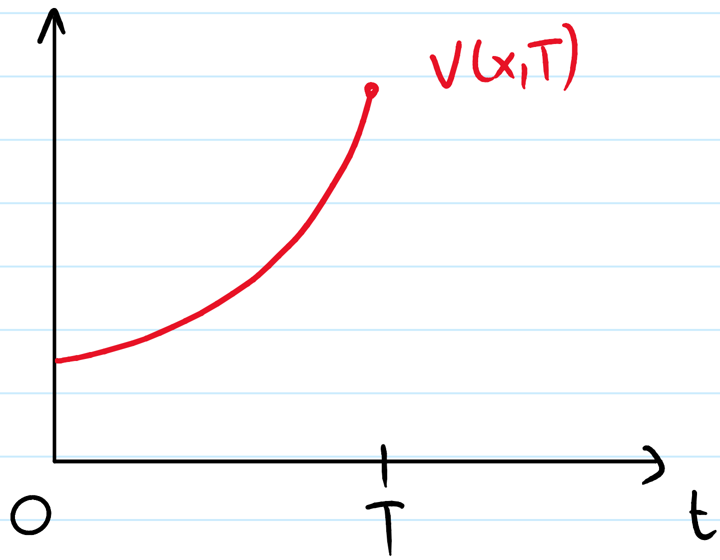
Picture:



In that case, if v attains its maximum at (x,T), then we still have $v_{xx} \leq 0$, but this time we only have $v_t \geq 0$, since v might increase until $t = T$

Picture:

(x fixed)



But still, in this case we still get the same contradiction, since we still have $v_t - v_{xx} \geq 0$

STEP 4: Hence the maximum of v is the larger one of:

$\max v(x,0) = \max \phi(x) + \varepsilon x^2 \leq \max \phi(x) + \max \varepsilon x^2 = (\max \phi(x)) + \varepsilon l^2$
$\max v(0,t) = \max g(t) + 0 = \max g(t)$
$\max v(l,t) = \max h(t) + \varepsilon l^2 = (\max h(t)) + \varepsilon l^2$

Therefore, we get that for all (x,t)

$v(x,t) \leq$ the larger one of: $(\max \phi(x)) + \varepsilon l^2$, $\max g(t)$, $(\max h(t)) + \varepsilon l^2$

(Notice that the right-hand-side is independent of x and t)

STEP 5: Finally, letting $\varepsilon \to 0$ in both sides of the above inequality and using $v = u + \varepsilon x^2 \to u$ (as $\varepsilon \to 0$), we get:

$u(x,t) \leq$ the larger one of $\max \phi(x)$, $\max g(t)$, $\max h(t)$

Note: The right-hand-side of the inequality basically represents the max on the bottom and lateral sides of the rectangle.

Since this holds for every (x,t), we get

$\max u \leq$ the larger one of $\max \phi(x)$, $\max g(t)$, $\max h(t)$

And therefore the max has to be attained at the bottom or the lateral sides of the rectangle, else we would get $\max u >$ the larger one of $\max \phi(x)$, $\max g(t)$, $\max h(t)$, which is a contradiction.