LECTURE 13: HEAT VS WAVE EQUATION

Wednesday, October 23, 2019 7:02 PM

Welcome to the half-time show of the course! So far we studied two important PDE: The heat equation and the wave equation, and today will just be a (cultural) overview of their similarities and differences.

I- REVIEW: WAVE AND HEAT

A) WAVE

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

$$\boxed{u(x,t) = 1/2\,(\phi(x-ct) + f(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi(s)\, ds}$$ (memorize)

B) HEAT EQUATION

$$\begin{cases} u_t = k\, u_{xx} \\ u(x,0) = \phi(x) \end{cases}$$

A solution is:

$$u(x,t) = S * \phi = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\,\phi(y)\,dy$$

Notice how different those two derivations are!

The rest of today is comparing properties of heat and wave equation

II- EXISTENCE, UNIQUENESS, STABILITY

A) WAVE

1) <u>Existence</u>: Yes (by D'Alembert's)

2) <u>Uniqueness</u>: Yes

Why?

First of all, for the wave equation, we haven't made any assumptions about the special form of our function (Compare with heat: We assumed $u(x,t) = 1/t^a\, v(\ldots)$), and all the steps for the derivation of D'Alembert's formula are in fact reversible.

Also: by Energy method: We multiplied $u_{tt} = u_{xx}$ (assume $c = 1$) by u_t and integrated by parts, to get that the energy

$$\boxed{E(t) = 1/2 \int_{-\infty}^{\infty} (u_t)^2 + (u_x)^2 \, dx}$$

is constant, from which we deduced uniqueness

3) Stability: Yes in an "integral" sense (by energy methods), but no in a "max" sense (no max principle, see below)

B) HEAT

1) Existence: Yes (by Fundamental Solution)

2) Uniqueness: Yes for finite rod ($0 < x < l$) by energy method and maximum principle. Also "Yes" for infinite rod, among solutions with the property $u(x,t) \le C\exp(ax^2)$

3) Stability: Yes for $0 < x < l$ (by energy method/maximum principle), "Yes" for infinite rod

Note: Here, for stability, we got:

$$\int_{-\infty}^{\infty} \big(U(x,t) - V(x,t)\big)^2 dx \le \int_{-\infty}^{\infty} \big(\phi_1(x) - \phi_2(x)\big)^2 dx$$

$$\text{MAX} \left| U(x,t) - V(x,t) \right| \le \text{MAX} \left| \phi_1(x) - \phi_2(x) \right|$$

Both versions say that if the initial conditions ϕ_1 and ϕ_2 are close enough, then the solutions u and v are close too (kind of like an epsilon-delta proof!)

Example: Suppose ϕ_1 and ϕ_2 are close in the sense $\max |\phi_1(x) - \phi_2(x)|$ is small (worst-case error is small). Then this says: $\max |u - v|$ is also small, that is u and v are close too!

III- MAXIMUM PRINCIPLE

As mentioned above, although the heat equation has a maximum principle, the wave equation does NOT have a maximum principle!

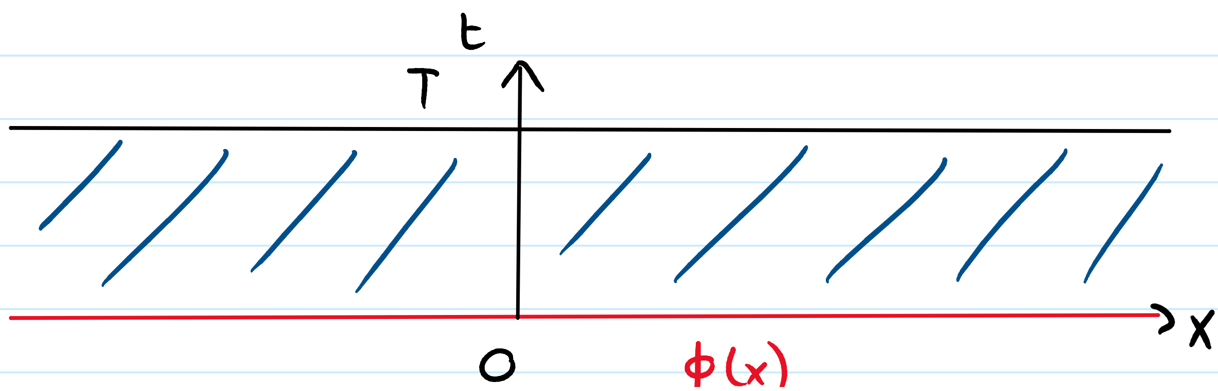
WAVE

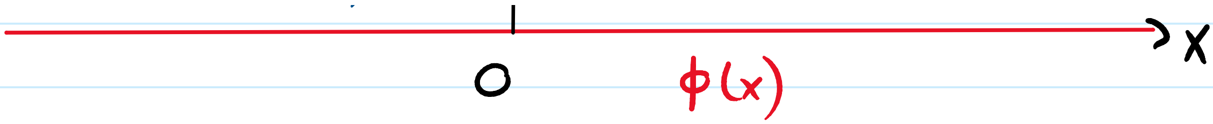
Note: Since there are no boundary conditions, maximum principle would mean:

Suppose $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \qquad (\text{and } u_t(x,0) = \psi(x)) \end{cases}$

Then $\max u = \max \phi(x)$ and $\min u = \min \phi(x)$

Picture: Here the rectangle is an infinite strip!





BUT: Let $\phi(x) = 0$ and $\psi(x) = 1$

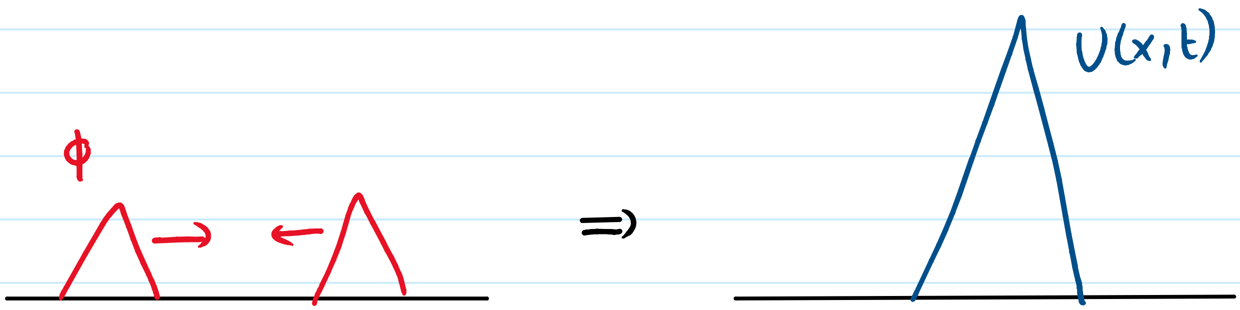
The maximum principle would say that max $u = 0$ and min $u = 0$, so $u = 0$

But D'Alembert gives:

$$u(x,t) = 1/2\,(0 + 0) + 1/(2c) \int_{x-ct}^{x+ct} 1\, ds = 1/(2c)\,(x + ct - (x - ct)) = t > 0$$

And even without that, notice that in the wave equation solutions can sometimes get bigger than the initial condition! (think resonance effect)

Picture: Although ϕ might be small, u might be huge (after collision)



IV- SMOOTHNESS

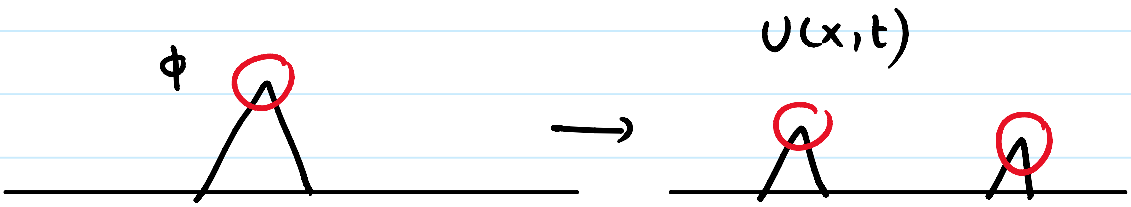
A) WAVE

Notice that in D'Alembert's formula,

$$u(x,t) = 1/2\,(\phi(x-ct) + \phi(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi(s)\, ds$$

u will never be smoother than ϕ, meaning that if ϕ has a corner somewhere, that corner won't disappear and will be transported (along the lines x - ct and x + ct)

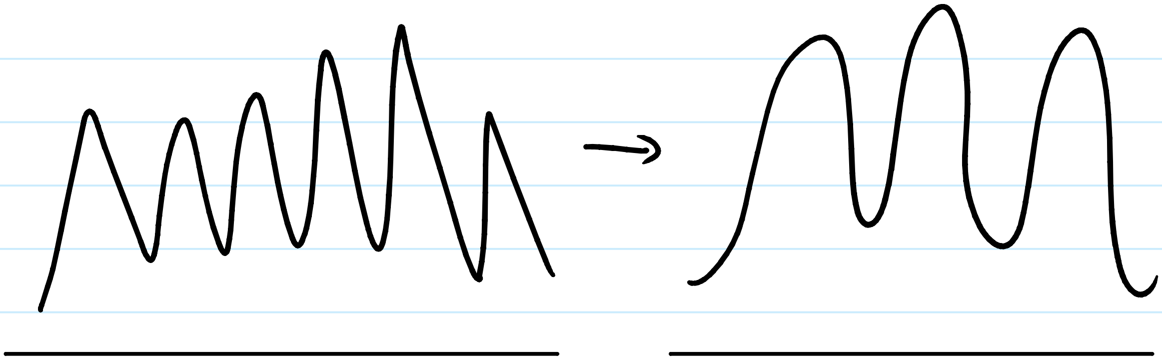
Picture:



B) HEAT

BUT for the heat equation, even if ϕ is VERY wild and non-differentiable, u(x,t) (at least the one given by the fundamental solution) will always be **INFINITELY** differentiable!!!

Picture: ϕ (wild) u(x,t) (smooth)



$-x^2$

Why? Notice $S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$ is infinitely differentiable

Then if $u = S * \phi$, then $u_x = (S * \phi)_x = \int_{-\infty}^{\infty} S(x-y)\,\phi(y)\,dy = \int_{-\infty}^{\infty} S_x(x-y)\,\phi(y)\,dy$ $= S_x * \phi$

And $u_{xx} = S_{xx} * \phi$, $u_{xxx} = S_{xxx} * \phi$, and similarly for u_t, etc.

Notice all the derivatives go on S, which is smooth, so in this sense we can calculate derivatives of u of all orders, so u is smooth too!

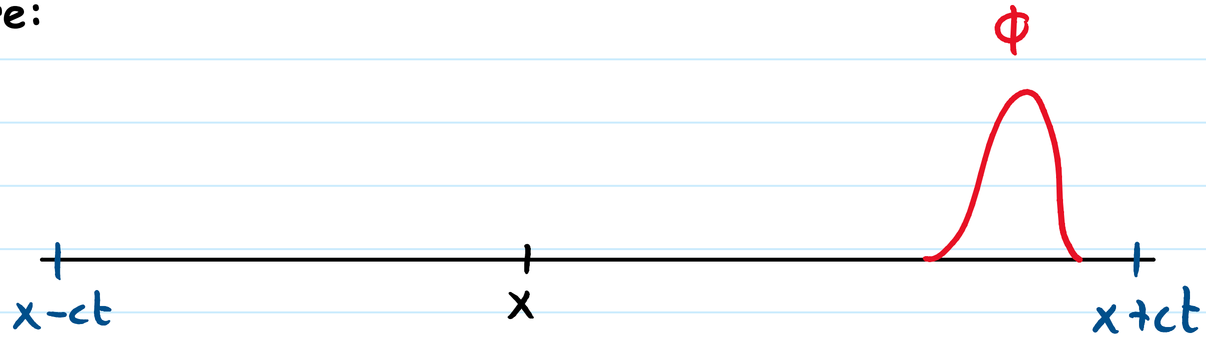
(In fact, that's what you notice in simulations of the heat equation: Pointy icebergs immediately melt to become a smoother surface)

V- SPEED OF PROPAGATION

A) WAVE

Recall: The wave equation has **FINITE** speed of propagation: Waves travel at speed at most c and it takes some time to feel the effect of the initial condition ϕ

Picture:



Analogy: If an alien lightyears away makes a sound, it will take some time for you to hear that sound

B) HEAT

BUT the heat equation has **infinite** speed of propagation!

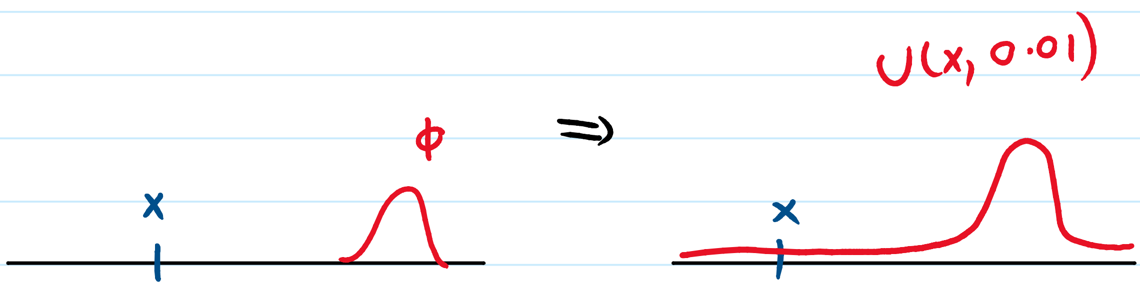
Why? Suppose $\phi = 0$ but somewhere **FAR** away it is > 0

Then $u(x,t) = S * \phi = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \underbrace{e^{\frac{-(x-y)^2}{4kt}}}_{>0} \underbrace{\phi(y)}_{>0} \, dy > 0$ (at least somewhere)

becomes IMMEDIATELY positive, even for small t !

This means you IMMEDIATELY feel the effect of the initial condition ϕ

Picture:



Analogy: If an alien lightyears away lights a fire, you will immediately feel the warmth, even if the effect is minimal.

VI- BACKWARD EQUATIONS

What if, in both equations, you go *backwards* in time?

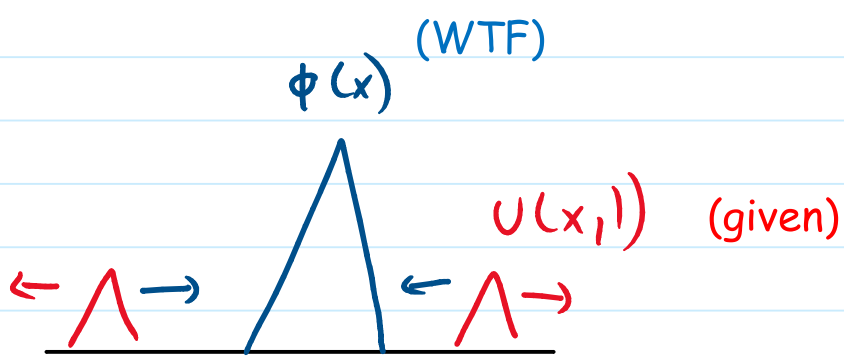
In other words:

Say t = 1, and I tell you what $u(x,1)$ is. Can you figure out what $u(x,0) = \phi(x)$ is?

A) WAVE

Yes for Wave Equation: Simply make the wave travel to the left instead of the right

Picture:



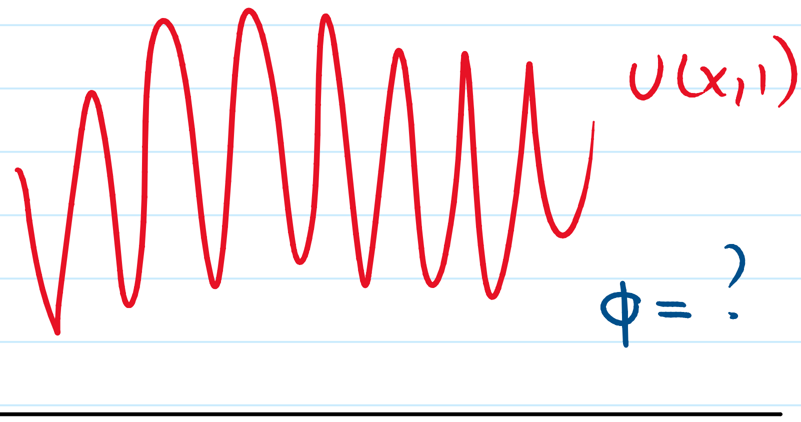
Interpretation: For the wave equation, information is preserved/transported

B) HEAT

BUT CANNOT do that for the heat equation! In fact, **CANNOT** go back in time for the heat equation!

Why? Suppose, $u(x,1)$ is given and is **VERY** wild/non differentiable

Picture:



Can you figure out what the initial profile $\phi(x)$ is?

In fact ϕ cannot exist, because if it existed, then by smoothness $u(x,t)$ would be differentiable, so $u(x,1)$ would be differentiable as well, but we assumed it is wild! (in other words, how can an iceberg melt to give you something wild?)

In fact, heat is known to be an irreversible process!

Interpretation: For the heat equation, information is lost (cannot figure out what caused a certain profile)

VII- LONG-TIME BEHAVIOR

What happens to $u(x,t)$ as t goes to infinity?

A) WAVE

The energy $E(t)$ is conserved, where $(c = 1)$:

$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (U_t)^2 + (U_x)^2 \, dx$

So u(x,t) cannot possibly go to 0 (at least in an integral sense)

In fact, compare with D'Alembert:

$$U(x,t) = 1/2\,(\phi(x-ct) + \phi(x+ct)) + 1/(2c) \int_{x-ct}^{x+ct} \psi(s)\, ds$$

The ϕ terms *might* eventually be 0, but the integral term isn't (if ψ is positive).

B) HEAT

u(x,t) goes to 0 as t goes to infinity

Why?

$$u(x,t) = S * \phi = \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\, dy$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{e^{\frac{-(x-y)^2}{4Kt}}}{\sqrt{4\pi K t}}}_{\to 0 \text{ As } t \to \infty} \phi(y)\, dy \quad \text{which goes to 0}$$

In fact, the energy

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} U^2 \, dx$$

Is decreasing, so there "should" be at least some decay