LECTURE 14: MIDTERM REVIEW SESSION

Friday, October 25, 2019 6:20 PM

I- ENERGY METHOD

Example: Use energy methods to show that the only solution of the following PDE is $u(x,t) = 0$

$$\begin{cases} u_t = k\, u_{xx} \\ u(0,t) = 0,\ u(l,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Hint: Multiply by u

1) $u_t\, u = k\, u_{xx}\, u$

$$\int_0^\ell u_t\, u\, dx = k \int_0^\ell u_{xx}\, u\, dx$$

2) Left-hand-side

$$\int_0^\ell u_t\, u\, dx = \int_0^\ell 1/2\ d/dt\ (u^2)\, dx = d/dt\ 1/2 \int_0^\ell u^2\, dx$$

3) Right-hand-side: Integrate by parts with x

$$u_{xx}\, u\, dx = u_x(l,t)\, \overset{0}{\cancel{u(l,t)}} - u_x(0,t)\, \overset{0}{\cancel{u(0,t)}} - \int_0^{\ell} u_x\, u_x\, dx \;\; = -\int_0^{\ell} (ux)^2\, dx$$

4) Equating both sides, we get

$$d/dt\ 1/2 \int_0^{\ell} u^2\ dx \ \leq\ -k \int_0^{\ell} (u_x)^2\, dx \ \leq 0$$

$$E'(t) \leq 0$$

5) Hence E(t) is decreasing, and in particular

$$E(t) \leq E(0)$$

$$(0 \leq)\ 1/2 \int_0^{\ell} u^2(x,t)\, dx \ \leq\ 1/2 \int_0^{\ell} \underbrace{u^2(x,0)}_{0}\, dx = 0$$

Hence $u^2(x,t) = 0$, so $u(x,t) = 0$

II- MAXIMUM PRINCIPLE

Example: Same problem, but with the maximum principle!

By max principle, max u is the larger one of

max u(0,t) = max 0 = 0
max u(l,t) = max 0 = 0
max u(x,0) = max 0 = 0

Hence max $u \le 0$, so $u(x,t) \le 0$

Similarly, by the minimum principle, min u is the smaller one of

$\min u(0,t) = \min 0 = 0$
$\min u(l,t) = \min 0 = 0$
$\min u(x,0) = \min 0 = 0$

Hence min $u \ge 0$, so $u(x,t) \ge 0$

Combining both, we get $u(x,t) = 0$

III- CHEN LU!

Example: Use the coordinate method to solve the PDE

$$a\, u_x + b\, u_y + cu = 0$$

Hint:

$$\begin{cases} \xi = ax + by \\ \eta = -bx + ay \end{cases}$$

$$u_x = \frac{\partial U}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= U_\xi\,(a) + U_\eta\,(-b)$$

$$= a\,U_\xi - b\,U_\eta$$

$$u_y = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= U_\zeta\,(b) + U_\eta\,(a)$$

$$= b\,U_\zeta + a\,U_\eta$$

So $a\,u_x + b\,u_y + cu = 0$

$$\Rightarrow a\,(a U_\zeta - b U_\eta) + b\,(b U_\zeta + a U_\eta) + cU = 0$$

$$\Rightarrow a^2 U_\zeta \cancel{- ab\,U_\eta} + b^2 U_\zeta \cancel{+ ab\,U_\eta} + cU = 0$$

$$\Rightarrow (a^2+b^2)\,U_\zeta + cU = 0$$

$$\Rightarrow U_\zeta = -\frac{c}{a^2+b^2}\,U$$

(Note: $y' = ky \Rightarrow y = Ce^{kt}$)

$$\Rightarrow U = f(\eta)\, e^{-\frac{c}{a^2+b^2}\zeta}$$

$$\Rightarrow \boxed{U(x,y) = f(ay - bx)\, e^{-\frac{c}{a^2+b^2}(ax+by)}}$$

IV- TRANSFORMS

Example: Now solve $a\,u_x + b\,u_y + cu = 0$

By using $v(x,y) = u(x,y)\, e^{(c/a)\,x}$

$\Rightarrow u = e^{-(c/a)x}\, v$

$$U_x = \left(e^{-\frac{cx}{a}} V\right)_x = -\frac{c}{a} e^{-\frac{cx}{a}} V + e^{-\frac{cx}{a}} V_x$$

$$U_y = \left(e^{-\frac{cx}{a}} V\right)_y = e^{-\frac{cx}{a}} V_y$$

$a\,u_x + b\,u_y + cu = 0$

$$\Rightarrow \quad a\left[-\frac{c}{a} e^{-\frac{cx}{a}} V + e^{-\frac{cx}{a}} V_x\right]$$

$$+ b\, e^{-\frac{cx}{a}} V_y + c\, \underbrace{e^{-\frac{cx}{a}} V}_{U} = 0$$

$$\Rightarrow \quad \cancel{-c\, e^{-\frac{cx}{a}} V} + a\, e^{-\frac{cx}{a}} V_x + b\, e^{-\frac{cx}{a}} V_y$$

$$\cancel{+ c\, e^{-\frac{cx}{a}} V} = 0$$

$-\cancel{cx}$

$$\Rightarrow \quad \cancel{e^{-\frac{cx}{a}}}\left[a V_x + b V_y\right] = 0$$

$$\Rightarrow \quad a V_x + b V_y = 0$$

$$\Rightarrow \quad V = f(ay - bx)$$

$$\Rightarrow \quad u e^{\frac{cx}{a}} = f(ay - bx)$$

$$\Rightarrow \quad \boxed{u(x,y) = f(ay - bx)\, e^{-(c/a)x}}$$

(Might look different from before, but basically the same if you write x in terms of the variables ax + by and ay - bx and use f arbitrary)

V- D'ALEMBERT

Example:

The general solution of $u_{xx} + 2\, u_{xt} - u_{tt} = 0$ is

$u(x,t) = F(2x-t) + G(x+t)$ (see Practice exam)

Find the solution that satisfies $u(x,0) = x^2$ and $u_t(x,0) = \sin(x)$

1) $u(x,0) = F(2x-0) + G(x+0) = F(2x) + G(x) = x^2$

$u_t(x,t) = -F'(2x-t) + G'(x+t)$
$u_t(x,0) = -F'(2x) + G'(x) = \sin(x)$

$\Rightarrow -F'(2x) + G'(x) = \sin(x)$ ⚠ $F'(2x) \neq (F(2x))'$
$\Rightarrow (-1/2\, F(2x))' + G(x) = \sin(x)$
$\Rightarrow -1/2\, F(2x) + G(x) = -\cos(x) + C$

$$\begin{cases} F(2x) + G(x) = x^2 \\ -1/2\, F(2x) + G(x) = -\cos(x) + C \end{cases}$$

2) Subtract both equations

$3/2\, F(2x) = x^2 + \cos(x) - C$

$F(2x) = 2/3\, x^2 + 2/3 \cos(x) - 2/3\, C$

$F(x) = 2/3\, (x/2)^2 + 2/3 \cos(x/2) - 2/3\, C$
$\quad = 1/6\, x^2 + 2/3 \cos(x/2) - 2/3\, C$

3) Add 2 times Equation 2 to Equation 1

$3\, G(x) = x^2 - 2\cos(x) + 2C$

$G(x) = 1/3\, x^2 - 2/3 \cos(x) + 2/3\, C$

4) Answer:

$u(x,t) = F(2x-t) + G(x+t)$
$\quad = 1/6\, (2x-t)^2 + 2/3 \cos((2x-t)/2) - 2/3\, C$
$\quad + 1/3\, (x+t)^2 - 2/3 \cos(x+t) + 2/3\, C$

$+ 1/3\,(x+t)^2 - 2/3\cos(x+t) + 2/3\,C$

$$\boxed{u(x,t) = 1/6\,(2x-t)^2 + 2/3\cos((2x-t)/2) - 2/3\cos(x+t)}$$

(No need to simplify)

VI- FIRST-ORDER PDE

Example: Solve $(1+x^2)\,u_x + e^y\,u_y = 0$

$$\frac{dy}{dx} = \frac{e^y}{1+x^2} \qquad \text{(= Slope)}$$

$$\Leftrightarrow (1+x^2)\,dy = e^y\,dx$$

$$\Leftrightarrow \int \frac{dy}{e^y} = \int \frac{dx}{1+x^2}$$

$$\Leftrightarrow \int e^{-y}\,dy = \int \frac{1}{1+x^2}\,dx$$

$$\Leftrightarrow -e^{-y} = \mathrm{TAN}^{-1}(x) + C$$

$$\Leftrightarrow C = \underbrace{-e^{-y} - \mathrm{TAN}^{-1}(x)}_{?}$$

Solution:

$$\boxed{u(x,y) = f(?) = f(-e^{-y} - \tan^{-1}(x)) \quad (= f(e^{-y} + \tan^{-1}(x))\,)}$$