LECTURE 10 - IMT + SUBSPACES

Wednesday, October 16, 2019 12:27 PM

I- IMT (continued)

Last time: Talked about the Invertible Matrix Theorem, which basically tells us when a matrix are invertible, and that invertible matrices are awesome.

Example: Is the following matrix invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

3 pivots, so YES

Example: If A is 3 x 3 and the columns of A span R^3, are they linearly independent?

YES

Columns of A span R^3
=> A is invertible (by IMT)
=> Columns of A are LI (by IMT)

AMAZING, because Span and LI are two different concepts, but for SQUARE matrices, they're the same!

Example: If A is 4 x 4 and $A\mathbf{x} = \mathbf{0}$ has a nonzero solution, does $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b}$?

NO

$(Ax = 0 \not\Rightarrow x = 0)$

$\Rightarrow$ A is not invertible (by IMT)

$\Rightarrow$ $Ax = b$ is not always consistent

II- NON-SQUARE MATRICES (Section 2.3)

WARNING: In the IMT, A must be $n \times n$!!!

For non-square matrices, the IMT is FALSE!

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then:

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So $AB = I$ $\quad$ ($\not\Rightarrow$ A is invertible)

BUT:

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$$

So $BA \neq I$, so A is NOT invertible!

In fact, let me show you that non-square matrices are **NEVER** invertible.

Example: Could a 2 x 3 matrix be invertible? (Columns > Rows)
NEVER !

$$A = \begin{array}{c} \begin{array}{ccc} & & \text{Free var} \\ & & \downarrow \end{array} \\ \begin{bmatrix} * & & \\ & * & \end{bmatrix} \end{array}$$

A has at most 2 pivots (< 3)
=> At least 1 free variable
=> ($Ax = 0 \not\Rightarrow x = 0$)
=> A is **NOT** invertible (Why? see # 23 in 2.1)

Example: Could a 3 x 2 matrix be invertible? (Rows > Columns)
NEVER!

$$A = \begin{bmatrix} * & \\ & * \\ 0 & 0 \end{bmatrix}$$

A has at most 2 pivots (< 3)
=> A doesn't have a pivot in every row
=> $Ax = b$ is **not** always consistent (by Row Theorem)
=> A is not invertible (see # 24 in 2.1)

Therefore, Invertible matrices must be SQUARE

III- SUBSPACES (Section 2.8)

WARNING: Sections 2.8 and 2.9 are VERY difficult and abstract!

GOAL: What sort of "info" does a matrix A give us?

To do this, we'll need to define the notion of a subspace (which is the *framework* we'll be dealing with, just like a frame of reference in physics)

Definition: A Subspace of R^n is any set H in R^n such that:

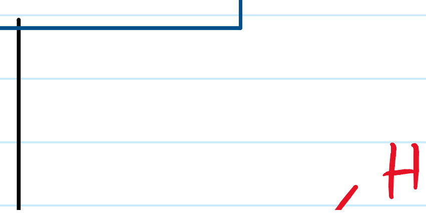
1. $\mathbf{0}$ is in H
2. If $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H
3. If $\mathbf{u}$ is in H, then $c\,\mathbf{u}$ is in H (for any number c)

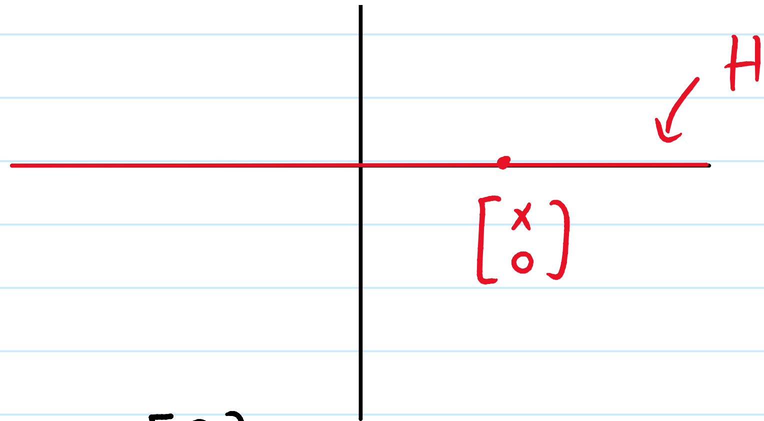
So subspaces are special regions in R^n that satisfy those 3 properties!

(Compare this with linear transformations, similar properties)

Ex 1: H = x-axis in R^2

$$H = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \ x \text{ is in } R \right\}$$





1) Is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in H?

Yes, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ with $x = 0$

2) Suppose $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} y \\ 0 \end{bmatrix}$ are in H

Then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \end{bmatrix}$ is in H

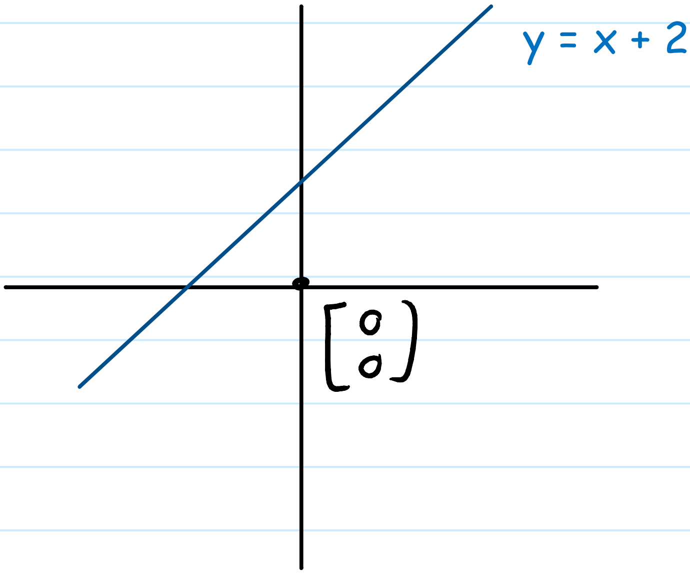
3) Suppose $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ is in H

Then $c\,\mathbf{u} = c \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix}$ is in H

Ex 2: $H = \{\mathbf{0}\}$

Ex 3: $H = R^n$

(Non) Ex 4: H = Line $y = x+2$ in R^2

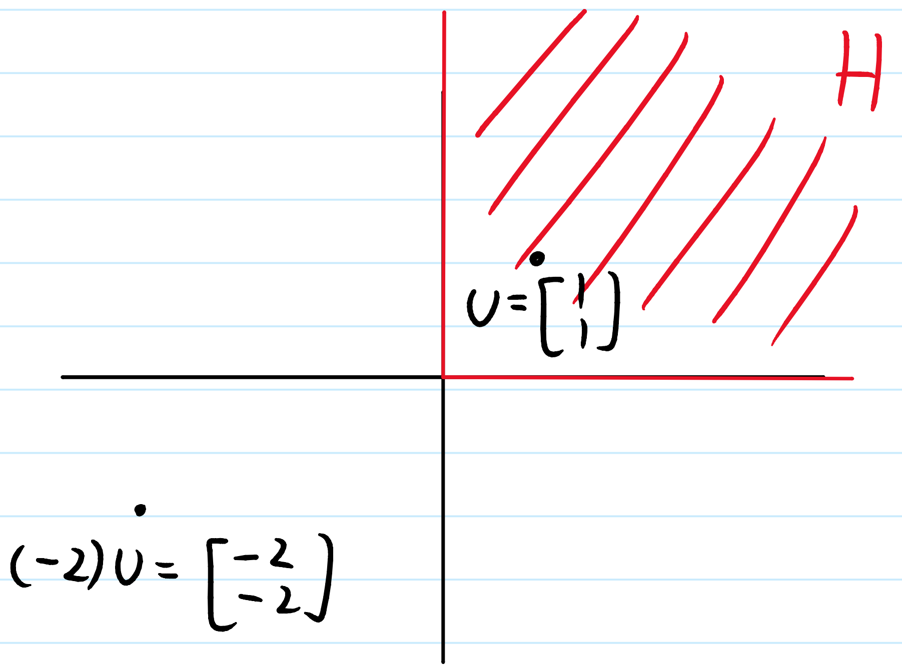


NO, $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in it

(Non) Ex 5:

H = First quadrant in R^2

$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \quad x \geq 0 \ \& \ y \geq 0 \right\}$$



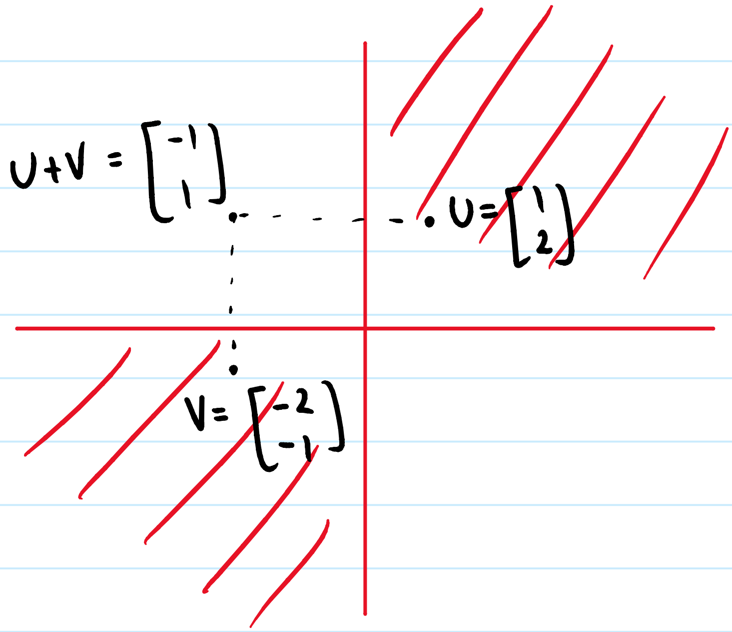
3) **u** = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in H

But (-2) **u** = $(-2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is NOT in H

So NOT a subspace

(Non) Ex 6: H = First and Third Quadrant in R^2

$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \quad xy \geq 0 \right\}$$



NO: 2) $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in H because (1)(2) = 2 $\geq$ 0

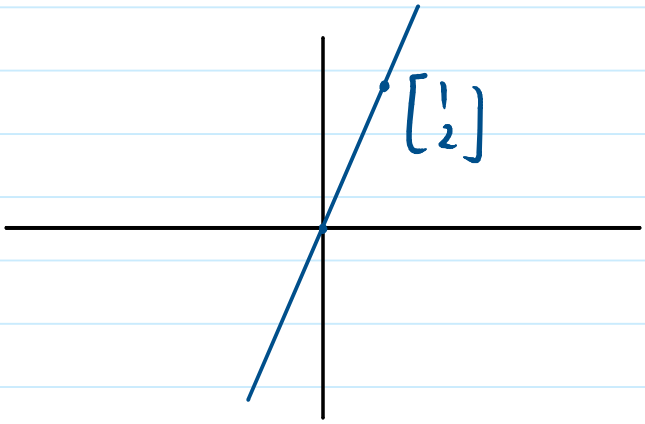
$\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ is in H because (-2)(-1) = 2 $\geq$ 0

But $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is not in H because (-1)(1) = -1 < 0

Luckily, we never have to check that H is a subspace, because we have the following super useful fact:

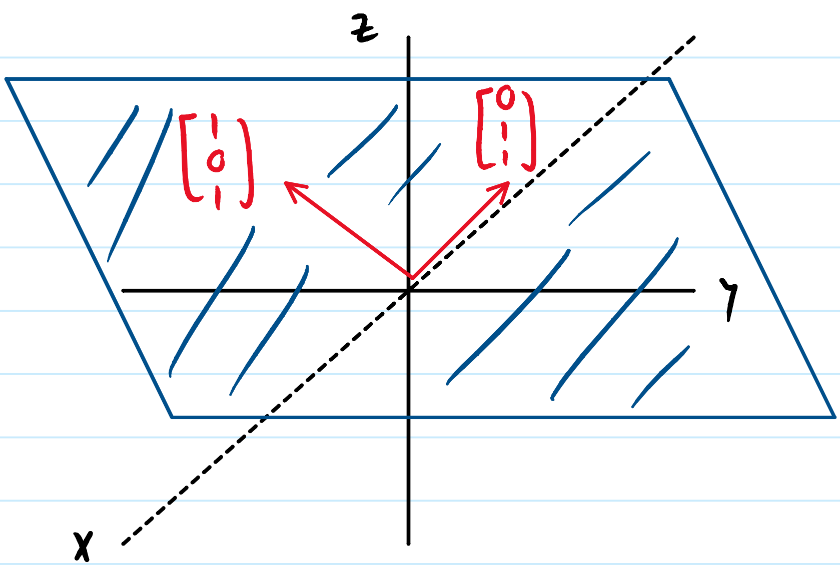
FACT: The Span of anything is always a subspace!

Ex 7: Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ = Line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Ex 8: Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$=$ Plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$



So typically, subspaces look like lines and planes