LECTURE 11: NUL(A) AND COL(A)

Friday, October 18, 2019 3:20 PM

Let's continue our subspace extravaganza! Our next task is to find an nice way of describing subspaces that will be useful below, which leads to the concept of a basis:

I- BASIS

Definition: A **basis** for H is a LI set whose span is H

Ex: Is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ a basis for R^3 ?

1) LI ? Yes
2) Span = R^3 ? Yes

So it's a basis, called the "Standard basis"

Ex: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is also a basis for R^3

(So could have many bases for the same space)

Ex: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ is not a basis for R^3 (LD, "too big")

Ex: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ is not a basis for R^3 (Span is not R^3, "too small")

So a basis is a way of describing a subspace with as few vectors as possible (kind of like a building block of subspaces)

II- NUL(A)

Goal: What "info" does a matrix A give us?

Turns out a matrix tells us two things:

1) How **bad** it is (= Nullspace)
2) How **good** it is (= Column Space)

Let's first start with how bad a matrix is!

Definition:

Nul(A) = Solutions of $A\mathbf{x} = 0$ = $\{\mathbf{x}$ such that $A\mathbf{x} = 0\}$

Example:

(a) Find Nul(A) where

$$A = \begin{bmatrix}1 & -2 & 1 & -5 & 6\\3 & -6 & 4 & -24 & 20\\2 & -4 & 0 & 8 & 1\end{bmatrix}$$

Solve $A\mathbf{x} = \mathbf{0}$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -5 & 6 & 0 \\ 3 & -6 & 4 & -24 & 20 & 0 \\ 2 & -4 & 0 & 8 & 1 & 0 \end{array}\right]$$

RREF $\longrightarrow$

$$\begin{array}{c} \begin{array}{cccccc} & y & & t & & \\ & \downarrow & & \downarrow & & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{cases} x - 2y + 4t = 0 \\ z - 9t = 0 \\ S = 0 \end{cases} \Rightarrow \begin{cases} x = 2y - 4t \\ z = 9t \\ s = 0 \end{cases}$$

$$x = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2y - 4t \\ y \\ 9t \\ t \\ 0 \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$

(y, t free)

$$\boxed{\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

LI!

FACT: This will always be a basis!

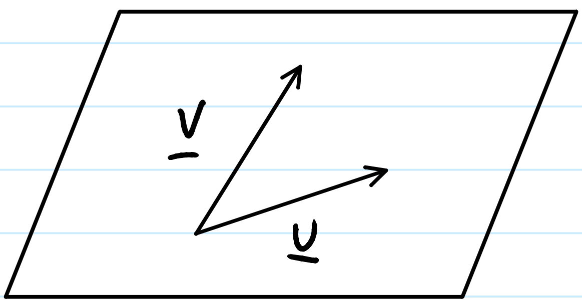
(b) Find a basis for Nul(A)

Answer: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\underline{u}$ $\underline{v}$

Interpretation: Nul(A) is a plane in R^5 (with directions **u** & **v**)

Nul(A) =



Remarks:

1) Nul(A) is a subspace of R^n

Here: A is 3 x 5 and Nul(A) is a subspace of R^5

2) Nul(A) deals with inputs, we're talking about x such that Ax = 0

3) Nul(A) measures how bad A is

Ex: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (BAD)

$\text{Nul}(A) = \mathbb{R}^3$ (because $Ax = 0$ for **every** x)

BIG

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (GOOD)

$\text{Nul}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ (because $Ax = 0 \Rightarrow x = 0$)

SMALL

The bigger the Nullspace, the worse the matrix

III- COLUMN SPACE

On the other hand, there's the column space, which measures how good a matrix is!

Definition: Col(A) = Span of columns of A

Ex: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Col(A) = Span $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$ $(= \textit{Span of columns of } A)$

Example: Find a basis for Col(A), where:

$$A = \begin{bmatrix}1 & -2 & 1 & -5 & 6\\ 3 & -6 & 4 & -24 & 20\\ 2 & -4 & 0 & 8 & 1\end{bmatrix}$$

REF

$$\rightarrow \begin{bmatrix}1 & 2 & 1 & -5 & 6\\ 0 & 0 & 1 & -9 & 2\\ 0 & 0 & 0 & 0 & -7\end{bmatrix}$$

Note: Pivots in columns 1, 3, 5

IMPORTANT: GO BACK TO A !!!

FACT: The pivot columns of A form a basis for Col(A)

Basis: $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}1\\4\\0\end{bmatrix}, \begin{bmatrix}6\\20\\1\end{bmatrix}\right\}$ (Columns 1, 3, 5 of A)

Says that Col(A) = Span $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}1\\4\\0\end{bmatrix}, \begin{bmatrix}6\\20\\1\end{bmatrix}\right\}$

Remarks:

1) Col(A) is a subspace of R^m (Column)

Here: A is 3 x 5, Col(A) is a subspace of R^3

2) Col(A) deals with outputs (mouthput):

b is in Col(A) <=> Ax = b has a solution

3) *Automatically* tells us that Columns 2 and 4 of A are LD

4) Col(A) measures how good a matrix is

Ex: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (BAD)

$$\text{Col}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\uparrow$

SMALL

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (GOOD)

$$\text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$= R^3$ <- BIG

The bigger the Column Space, the better the matrix

Note: Col(A) actually helps us find bases!

Example: Find a basis for

$$H = \text{Span}\left\{\begin{bmatrix}1\\-4\\-3\end{bmatrix}, \begin{bmatrix}-3\\6\\7\end{bmatrix}, \begin{bmatrix}-4\\-2\\6\end{bmatrix}\right\}$$

SAME as finding a basis for Col(A), where:

$$A = \begin{bmatrix}1 & -3 & -4\\-4 & 6 & -2\\-3 & 7 & 6\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix}1 & -3 & -4\\0 & -6 & -18\\0 & 0 & 0\end{bmatrix}$$

Basis for H: $\left\{\begin{bmatrix}1\\-4\\3\end{bmatrix}, \begin{bmatrix}-3\\6\\7\end{bmatrix}\right\}$

(So H is a plane in R^3)

Example: With A as above, Find a basis for Nul(A)

Continue row-reducing:

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & -6 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{cccc} & & z & \\ & & \downarrow & \end{array} \\ \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{cases} x + 5z = 0 \\ y + 3z = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x = -5z \\ y = -3z \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5z \\ -3z \\ z \end{bmatrix} = z \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

Basis for Nul(A): $\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}$