LECTURE 12: DIMENSION AND RANK

Monday, October 21, 2019 11:08 AM

I- DIMENSION

Today: All about dimension, which is the size of a subspace

Definition: dim(H) = Number of vectors in a basis of H

Ex: What is dim(R^3) ?

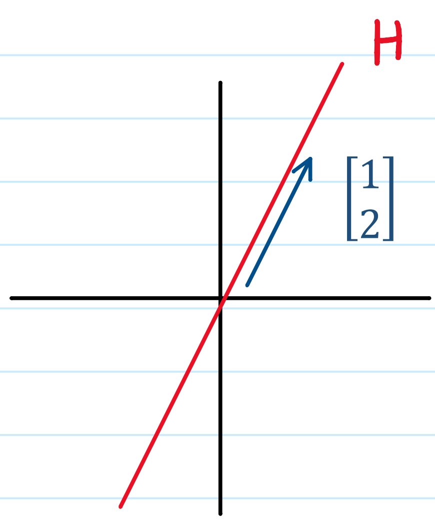
1) Find a basis for R^3 : $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2) Count the number of vectors in that basis:

Ans: 3

(Intuitively: R^3 has 3 'directions')

Ex: H = Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$



Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Dim = 1

(THIS is why lines are 1 dimensional, only have 1 direction)

Note: Dimensions are pretty neat!

Ex: 3 LI vectors in R^3 (dim = 3) automatically span R^3

II- RANK

Let's apply dimension to the concepts of Nul(A) and Col(A) discussed last time!

Example: (EXCELLENT Exam Question)

Let $A = \begin{bmatrix} 3 & 0 & 6 & 9 & 0 \\ 2 & 0 & 4 & 7 & 2 \\ 3 & 0 & 6 & 6 & -6 \end{bmatrix} \overset{REF}{\sim} \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find a basis for Col(A)

Pivots in Columns 1 and 3

Basis: $\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \right\}$ (2 vectors)

(b) Find dim(Col(A)) = Rank(A)

Ans: 2

$\boxed{\text{Definition: Rank(A) = Dim(Col(A)) = \# of Pivots}}$

Note:

1) The bigger Rank(A), the better A is

Ex: $A = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ (bad), Rank(A) = 0 (small)

$A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ (good), Rank(A) = 3 (big)

2) Rank(A) gives you the "true" size of A

Ex: $\begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}$ is 3 x 3

But has Rank = 1, so it really has size 1, only 1 piece of info

(c) Find a basis for Nul(A) (<- $Ax = 0$)

$$A \sim \begin{bmatrix}1&-2&5&4\\0&0&3&6\\0&0&0&0\end{bmatrix} \sim \begin{array}{c}\begin{array}{ccccc} & y & & t & \end{array}\\ \left[\begin{array}{cccc|c}1&-2&0&-6&0\\0&0&1&2&0\\0&0&0&0&0\end{array}\right]\end{array}$$

$$\begin{cases} x - 2y - 6t = 0 \\ z + 2t = 0 \end{cases} \Rightarrow \begin{cases} x = 2y + 6t \\ z = -2t \end{cases}$$

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y+6t \\ y \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6t \\ 0 \\ -2t \\ t \end{bmatrix}$$

$$= y\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

LI

Basis: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

(d) Find Dim(Nul(A))

Dim(Nul(A)) = 2 = # of free variables

III- RANK-NULLITY THEOREM

(e) Calculate:

Rank(A) + Dim(Nul(A))
= 2 + 2
= 4
= n (= # columns of A)

FACT: [RANK-NULLITY THEOREM]

$$\text{Rank}(A) + \text{Dim}(\text{Nul}(A)) = n \qquad (*)$$

Remarks:

1) n as in Nullity

2) Rank(A) = # Pivots
   Dim(Nul(A)) = # Free Variables

   So (*) just says: # Pivots + # Free variables = # Columns

3) Intuitively:

   Rank(A) measures how GOOD A is
   Dim(Nul(A)) measures how BAD A is

   So (*) says that they balance out, like the Yin Yang of Math!

IV- IMT DELUXE

Now that we know about Col and Rank, we can "upgrade" our IMT

THEOREM [IMT DELUXE]

Let A be n x n (Think $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$)

Then:

1) A is invertible

2) Rank(A) = n, Col(A) = R^n

3) Nul(A) = {**0**}, dim(Nul(A)) = 0

4) (The columns of A form a basis for R^n)

V- COORDINATES

Lastly, I'd like to cover a fun topic not related to rank, but related to bases. The real reason bases are so important is that they allow us to define coordinates

Example: Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$ be a basis for R^2

(a) Find **x** given $[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

"coordinates of **x** with respect to B"

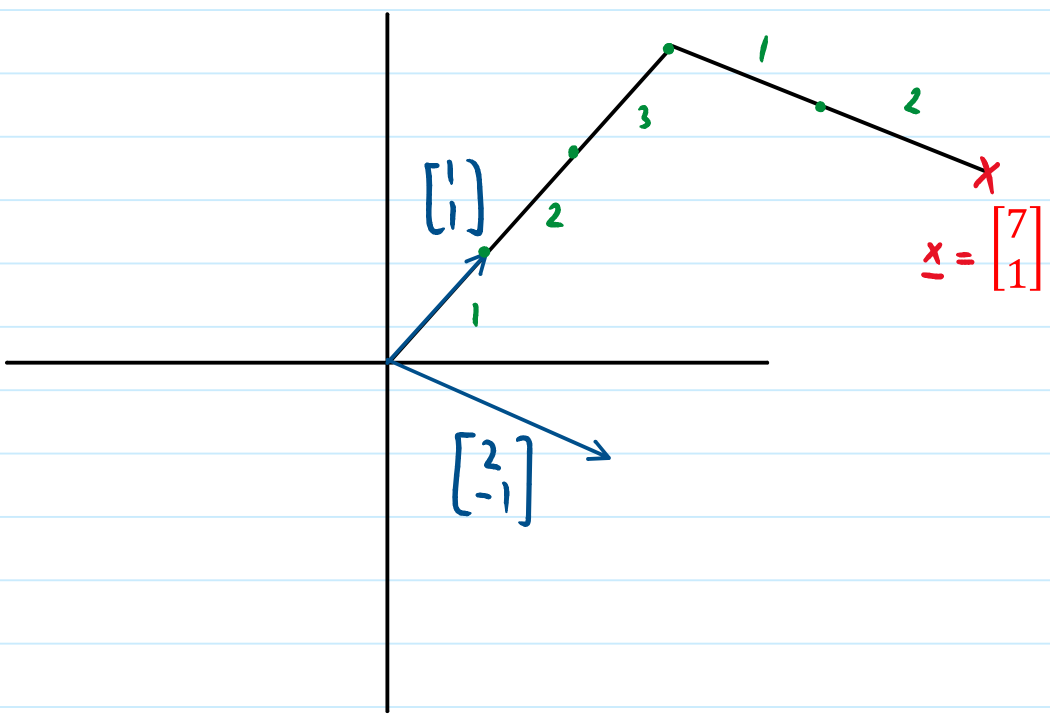
All this means is: To find **x**, you have to go 3 steps in the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ direction and 2 steps in the $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ direction (think of it as a treasure hunt)

Definition: $[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ means:

$$\underline{x} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$\mathcal{B}$

Picture: Here we're **given** the number of steps & we want to find **x**



(b) Find $[\mathbf{x}]_B$ given $\mathbf{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$\mathcal{B}$

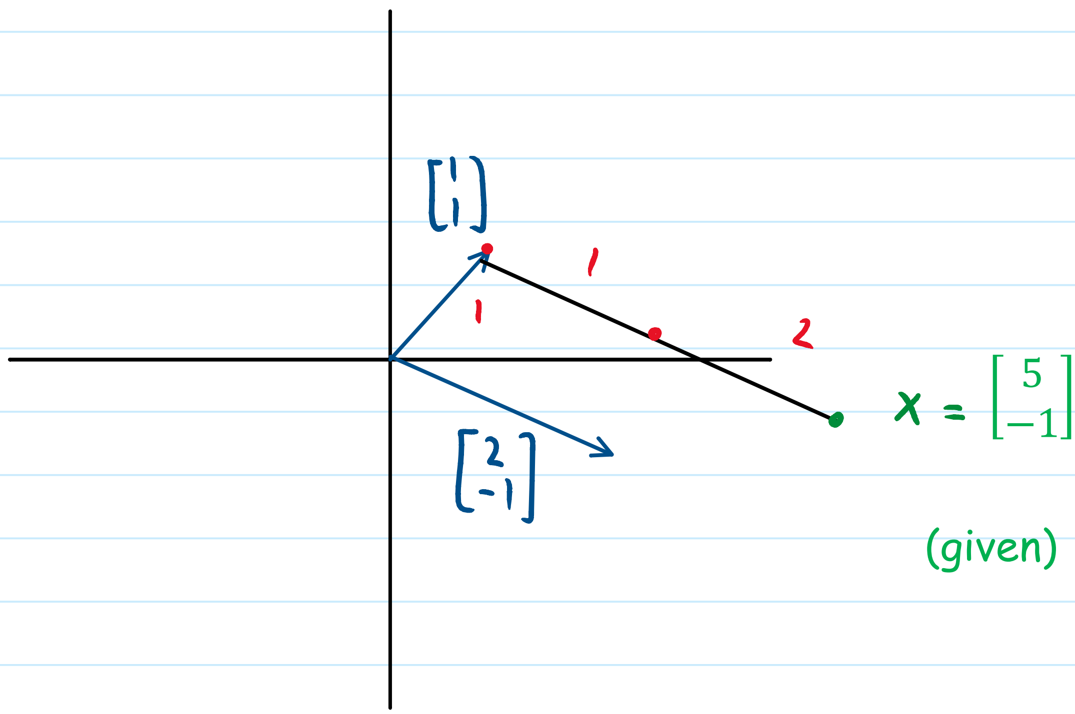
$[x]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$ means $a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \underline{x}$

(Want To Find = WTF) $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & -1 & -1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\Rightarrow \boxed{[x]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Picture: Here we're **given x** and we want to find the number of steps $[\mathbf{x}]_B$



(We'll come back to coordinates in section 5.4)

VI- CHANGE OF COORDINATES MATRIX (optional)

Matrix way of doing the previous problem
(faster and more awesome)

$\mathcal{B}$

Let P = $\begin{bmatrix}1 & 2\\1 & -1\end{bmatrix}$ (change of coordinates matrix)

And use:

$$\boxed{\text{FACT:} \quad P\,[\mathbf{x}]_B = \mathbf{x}}$$

(a) $[\mathbf{x}]_B = \begin{bmatrix}3\\2\end{bmatrix}$

(a) $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\mathbf{x} = P\,[\mathbf{x}]_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

(b) $\mathbf{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$\mathbf{x} = P[\mathbf{x}]_B$

$\Rightarrow [\mathbf{x}]_B = P^{-1}\mathbf{x}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Note: Think of $[\mathbf{x}]_B$ as a 'barcode' and P as a barcode scanner. Then $P[\mathbf{x}]_B = \mathbf{x}$ means that P scans the barcode $[\mathbf{x}]_B$ and tells you what the item $\mathbf{x}$ is