Welcome to my favorite math topic of all time: DETERMINANTS!!!

I- CALCULATING DETERMINANTS

Definition: \( \text{det}(A) = |A| = \) Some number associated to A

(The determinant is a magical number that gives us TONS of useful info about A. For example, we'll see today that it'll give us a 1 second way of determining if A is invertible)

How do we calculate determinants? Just like for inverses, we proceed step by step

**Ex:** \(|-7| = -7\) (so doesn't have to be \(\geq 0\))

**Ex:** \[
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix}
= ad - bc \quad \text{(similar to the formula of 2x2 inverses)}
\]

**Ex:**
\[
\begin{vmatrix}
  2 & 3 & 0 \\
  2 & 0 & 3 \\
  -1 & 1 & 1 \\
\end{vmatrix}
= ?
\]

**STEP 1:** Pick a row or column (usually one with lots of 0's)

Here: Row 1

**STEP 2:** SIGN TABLE: Start with + on the upper-left corner, and every time you jump, you change signs
This tells us:

\[
\begin{vmatrix}
2 & 3 & 0 \\
2 & 0 & 3 \\
-1 & 1 & 1 \\
\end{vmatrix}
= +2 (...) - 3(...) + 0(...) \\
\]

**STEP 3:** It’s Bomberman time!!!

Bomberman: Fun game where the character places a bomb, and the bomb destroys *everything* in the same row and column.

For example:

\[
\begin{vmatrix}
2 & 3 & 0 \\
2 & 0 & 3 \\
-1 & 1 & 1 \\
\end{vmatrix}
= \begin{vmatrix} 0 & 3 \\
0 & 1 \\
\end{vmatrix}
\]
And this is how you calculate determinants! For every number in the row/column you picked (here Row 1), you place a bomb at that number and you take the determinant of the rest!

\[
\begin{vmatrix}
2 & 3 & 0 \\
2 & 0 & 3 \\
-1 & 1 & 1 \\
\end{vmatrix} = +2 \begin{vmatrix}
0 & 3 \\
1 & 1 \\
\end{vmatrix} - 3 \begin{vmatrix}
2 & 3 \\
-1 & 1 \\
\end{vmatrix} + 0 \begin{vmatrix}
2 & 0 \\
-1 & 1 \\
\end{vmatrix}
\]

\[
= 2(0(1) - (3)(1)) - 3(2(1) - (3)(-1)) + 0(2(1) - (0)(-1))
\]

\[
= 2(-3) - 3(5)
\]

\[
= -21
\]

**Example:** Expand along Column 2

\[
\begin{vmatrix}
1 & 0 & -1 \\
3 & 0 & 5 \\
6 & 9 & 7 \\
\end{vmatrix} = (-0)\begin{vmatrix}
3 & 5 \\
6 & 7 \\
\end{vmatrix} + 0\begin{vmatrix}
1 & -1 \\
6 & 7 \\
\end{vmatrix} - 9\begin{vmatrix}
1 & -1 \\
3 & 5 \\
\end{vmatrix}
\]

\[
= -9(5 + 3)
\]

\[
= -72
\]

**NOTICE** how the 0's were helpful here! In fact, let's do a Bomberman Extravaganza!
Example:

\[
\begin{vmatrix}
\frac{1}{3} & 4 & 0 & \frac{1}{2} & 0 \\
\pi & 3 & 4 & 2 & 1 \\
\chi & 0 & 0 & 2 & 3 \\
9001 & 0 & 0 & 5 & 7 \\
\end{vmatrix}
= (+1) 
\begin{vmatrix}
4 & 0 & \frac{1}{2} & 0 & \infty \\
\pi & 3 & 4 & 2 & 1 \\
0 & 0 & 2 & 3 \\
0 & 0 & 5 & 7 \\
\end{vmatrix}
(\times 0) \ldots
\]

\[
= (-0) \ldots + 3 
\begin{vmatrix}
4 & \frac{1}{2} & \infty \\
0 & 2 & 3 \\
0 & 5 & 7 \\
\end{vmatrix}
\]

\[
= (3)(4) 
\begin{vmatrix}
2 & 3 \\
5 & 7 \\
\end{vmatrix}
\]

\[
= 12 \left(14 - 15\right)
\]

\[
= -12
\]

**II- DETERMINANTS AND ROW-REDUCTION**

You can also evaluate determinants using row-reduction (and in general that's how computers evaluate them because it's faster)
Rules:

1. Interchanging two rows puts a minus sign in front of det
2. Dividing by 2 puts a 2 in front of det ("You factor 2 out")
3. Adding a multiple of a row to another doesn't change det

(Amazing: The most important operation doesn't do anything to det!)

Example:

\[
\begin{vmatrix}
2 & 13 & -7 \\
1 & 5 & -3 \\
3 & -3 & 3 \\
\end{vmatrix} = 3
\begin{vmatrix}
2 & 13 & -7 \\
1 & 5 & -3 \\
1 & -1 & 1 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
2 & 13 & -7 \\
1 & 5 & -3 \\
1 & -1 & 1 \\
\end{vmatrix} = (x-2) \begin{vmatrix}
2 & 13 & -7 \\
0 & 3 & -1 \\
0 & -1 & 1 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & 5 & -3 \\
0 & 3 & -1 \\
0 & -1 & 1 \\
\end{vmatrix}
\]

(Does nothing)
\[
\begin{vmatrix}
0 & 3 & -1 \\
0 & -6 & 4 \\
\end{vmatrix} \div -2
\]

\[
= (-3)(-2) \begin{vmatrix}
1 & 5 & -3 \\
0 & 3 & -1 \\
0 & 3 & -2 \\
\end{vmatrix} \\
\]

\[
= 6 \begin{vmatrix}
1 & 5 & -3 \\
0 & 3 & -1 \\
0 & 0 & -1 \\
\end{vmatrix} \\
\]

\[
= 6 \cdot (1)(3)(-1) \\
= -18
\]

(Works REALLY well with larger matrices!)

III- PROPERTIES OF DETERMINANTS

Determinants have REALLY nice properties!

MY SINGLE FAVORITE PROPERTY:

**IMT DELUXE DELUXE**

A is invertible \( \iff \det(A) \neq 0 \)
A is invertible $\iff$ det(A) ≠ 0

(In other words, this gives us a 1 second way of checking if A is invertible)

Ex: \[
\begin{vmatrix}
2 & 13 & -7 \\
1 & 5 & 3 \\
3 & -3 & 3
\end{vmatrix} = -18 \neq 0 \quad \text{Invertible!}
\]

Ex: \[
\begin{vmatrix}
1 & 2 \\
2 & 4
\end{vmatrix} = 4 - 4 = 0 \quad \text{Not invertible}
\]

Other properties:

1) det(AB) = det(A) det(B)
2) det(A^{-1}) = 1/det(A)
3) det(A^T) = det(A)

Example: Suppose A = PDP^{-1} ("A is similar to D", Chapter 5)

Show det(A) = det(D)

\[
det(A) = det(PDP^{-1})
= det(P) det(D) det(P^{-1})
= det(P) det(D) (1/det(P))
= det(D)
\]

Example: Suppose Q^T Q = I ("Q is orthogonal", Chapter 6)
Show $\det(Q) = \pm 1$

$Q^T Q = I$

$\Rightarrow \det(Q^T Q) = \det(I) = 1$

$I =\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \det(Q^T) \det(Q) = 1$

$\Rightarrow \det(Q) \det(Q) = 1$

$\Rightarrow (\det(Q))^2 = 1$

$\Rightarrow \det(Q) = \pm 1$

WARNING:

$\det(A + B) \neq \det(A) + \det(B)$

$\det(cA) \neq c \det(A)$

(det is NOT linear!)

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$|A| + |B| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 + 0 = 0$

$|A + B| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $c = 3$
\[ c |A| = 3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 3(1) = 3 \]

\[ 1c |A| = \begin{vmatrix} 3 & 10 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9 \]