LECTURE 13: DETERMINANTS

Wednesday, October 23, 2019 12:16 PM

Welcome to my favorite math topic of all time: DETERMINANTS!!!

I- CALCULATING DETERMINANTS

Definition: det(A) = |A| = Some number associated to A

(The determinant is a magical number that gives us **TONS** of useful info about A. For example, we'll see today that it'll give us a 1 second way of determining if A is invertible)

How do we calculate determinants? Just like for inverses, we proceed step by step

Ex: $|[-7]| = -7$ (so doesn't have to be ≥ 0)

Ex: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (similar to the formula of 2x2 inverses)

Ex: $\begin{vmatrix} 2 & 3 & 0 \\ 2 & 0 & 3 \\ -1 & 1 & 1 \end{vmatrix} = ?$

STEP 1: Pick a row or column (usually one with lots of 0's)

Here: Row 1

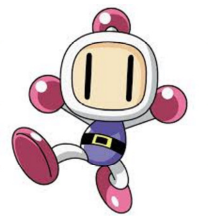
STEP 2: SIGN TABLE: Start with + on the upper-left corner, and every time you jump, you change signs

$$\begin{vmatrix} \boxed{+ \ - \ +} \\ - \ + \ - \\ + \ - \ + \end{vmatrix}$$

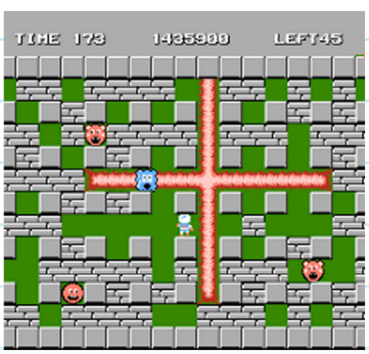
This tells us:

$$\begin{vmatrix} \boxed{2 \quad 3 \quad 0} \\ 2 \quad 0 \quad 3 \\ -1 \quad 1 \quad 1 \end{vmatrix} = +2\,(\ldots) - 3(\ldots) + 0(\ldots)$$

STEP 3: It's Bomberman time!!!



Bomberman: Fun game where the character places a bomb, and the bomb destroys *everything* in the same row and column.



For example:

$$\begin{vmatrix} 2 & 3 & 0 \\ 2 & 0 & 3 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix}$$

And this is how you calculate determinants! For every number in the row/column you picked (here Row 1), you place a bomb at that number and you take the determinant of the rest!

$$\begin{vmatrix} 2 & 3 & 0 \\ 2 & 0 & 3 \\ -1 & 1 & 1 \end{vmatrix} = +2\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix}$$

$= 2\big((0)(1) - (3)(1)\big) - 3\big((2)(1) - (3)(-1)\big)$
$+ 0\big((2)(1) - (0)(-1)\big)$
$= 2(-3) - 3(5)$

$= -21$

Example: Expand along Column 2

$$\begin{vmatrix} 1 & \boxed{0} & -1 \\ 3 & \boxed{0} & 5 \\ 6 & \boxed{9} & 7 \end{vmatrix} = (-0)\begin{vmatrix} 3 & 5 \\ 6 & 7 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 6 & 7 \end{vmatrix} - 9\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix}$$

$= -9(5+3)$
$= -72$

NOTICE how the 0's were helpful here! In fact, let's do a Bomberman Extravaganza!

Example:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ \sqrt{5} & 4 & 0 & \frac{1}{0} & \infty \\ \pi & e & 3 & 42 & i \\ \lambda & 0 & 0 & 2 & 3 \\ 9001 & 0 & 0 & 5 & 7 \end{vmatrix} = (+1) \begin{vmatrix} 4 & 0 & \frac{1}{0} & \infty \\ e & 3 & 42 & i \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 5 & 7 \end{vmatrix} (-0) \cdots$$

$$= (-0) \cdots \quad + 3 \begin{vmatrix} 4 & \frac{1}{0} & \infty \\ 0 & 2 & 3 \\ 0 & 5 & 7 \end{vmatrix}$$

$$= (3)(4) \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}$$

$$= 12\,(14 - 15)$$

$$= -12$$

II- DETERMINANTS AND ROW-REDUCTION

You can also evaluate determinants using row-reduction (and in general that's how computers evaluate them because it's faster)

Rules:

1. Interchanging two rows puts a minus sign in front of det
2. Dividing by 2 puts a 2 in front of det ("You factor 2 out")
3. Adding a multiple of a row to another doesn't change det

(Amazing: The most important operation doesn't do anything to det!)

Example:

$$\begin{array}{r} \\ \\ (\div 3) \end{array} \begin{vmatrix} 2 & 13 & -7 \\ 1 & 5 & -3 \\ 3 & -3 & 3 \end{vmatrix} = 3 \begin{vmatrix} 2 & 13 & -7 \\ 1 & 5 & -3 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= (-3) \begin{vmatrix} 1 & 5 & -3 \\ 2 & 13 & -7 \\ 1 & -1 & 1 \end{vmatrix} \begin{array}{l} \\ (\times -2) \\ (\times -1) \end{array}$$

(does nothing)

$$= (-3) \begin{vmatrix} 1 & 5 & -3 \\ 0 & 3 & -1 \\ 0 & -6 & 4 \end{vmatrix} \begin{array}{l} \\ \\ (\div -2) \end{array}$$

$$\begin{vmatrix} 0 & 3 & -1 \\ 0 & -6 & 4 \end{vmatrix} \quad (\div -2)$$

$$= (-3)(-2) \begin{vmatrix} 1 & 5 & -3 \\ 0 & 3 & -1 \\ 0 & 3 & -2 \end{vmatrix} \; (x-1)$$

$$= 6 \begin{vmatrix} 1 & 5 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= 6 \; (1)(3)(-1)$$

$$= -18$$

(Works REALLY well with larger matrices!)

III- PROPERTIES OF DETERMINANTS

Determinants have **REALLY** nice properties!

MY SINGLE FAVORITE PROPERTY:

IMT DELUXE DELUXE

A is invertible <=> det(A) ≠ 0

A is invertible <=> det(A) ≠ 0

(In other words, this gives us a 1 second way of checking if A is invertible)

Ex: $\begin{vmatrix} 2 & 13 & -7 \\ 1 & 5 & 3 \\ 3 & -3 & 3 \end{vmatrix} = -18 \neq 0$ Invertible!

Ex: $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$ Not invertible

Other properties:

1) $\det(AB) = \det(A)\det(B)$
2) $\det(A^{-1}) = 1/\det(A)$
3) $\det(A^T) = \det(A)$

Example: Suppose $A = PDP^{-1}$ ("A is similar to D", Chapter 5)

Show $\det(A) = \det(D)$

$$
\begin{aligned}
\det(A) &= \det(PDP^{-1}) \\
&= \det(P)\det(D)\det(P^{-1}) \\
&= \cancel{\det(P)}\det(D)(1/\cancel{\det(P)}) \\
&= \det(D)
\end{aligned}
$$

Example: Suppose $Q^T Q = I$ ("Q is orthogonal", Chapter 6)

Show $\det(Q) = \pm 1$

$Q^T Q = I$

$\Rightarrow \det(Q^T Q) = \det(I) = 1$

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\Rightarrow \det(Q^T)\det(Q) = 1$

$\Rightarrow \det(Q)\det(Q) = 1$

$\Rightarrow (\det(Q))^2 = 1$

$\Rightarrow \det(Q) = \pm 1$

WARNING:

$\det(A + B) \neq \det(A) + \det(B)$

$\det(cA) \neq c \det(A)$

(det is NOT linear!)

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$|A| + |B| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 + 0 = 0$$

$$|A + B| = \left| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad c = 3$

$$c\,|A| = 3\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 3(1) = 3$$

$$|cA| = \left| 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9$$