LECTURE 14: APPLICATIONS OF DETERMINANTS

Friday, October 25, 2019 8:01 PM

Today: All about applications of determinants, so you can see how awesome they really are!

I- CRAMER'S RULE

First of all, determinants allow us to solve systems of equations in a snap!

Example: Use Cramer's Rule to solve:

$$\begin{cases} -5x + 3y = 9 \\ 3x - y = -5 \end{cases}$$

$$\begin{bmatrix} -5 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

$$A \quad \mathbf{x} = \mathbf{b}$$

Cramer's Rule: |A| but with **b** in first column

$$x = \frac{\begin{vmatrix} 9 & 3 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{6}{-4} = -3/2$$

|A|

$$y = \frac{\begin{vmatrix} -5 & 9 \\ 3 & -5 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{-2}{-4} = -1/2$$

Answer:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$$ WOW!!!

(This gives us a 1 second way of solving a system, without using row-reduction! But in practice it's a pain, Ex: To solve a 3x3 system, have to find four 3x3 determinants)

II- INVERSES

Just like we can use determinants to solve systems, we can also use determinants to find the inverse of a matrix!

Example: Find A^{-1} where

$$A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$|A| = \ldots = -1$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} +\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} \\ +\begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^{T}$$

"Adjugate matrix"

(= matrix with +/- signs and all mini determinants.
Basically, you make the sign table and place bombs everywhere!)

$$\begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$= \frac{1}{-1} \begin{bmatrix} 5 & 2 & -4 \\ -3 & -2 & 3 \\ -8 & -3 & 6 \end{bmatrix}^T$$

$$= - \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

Example: Find A^{-1} where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = (1/|A|) \begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix}^T = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

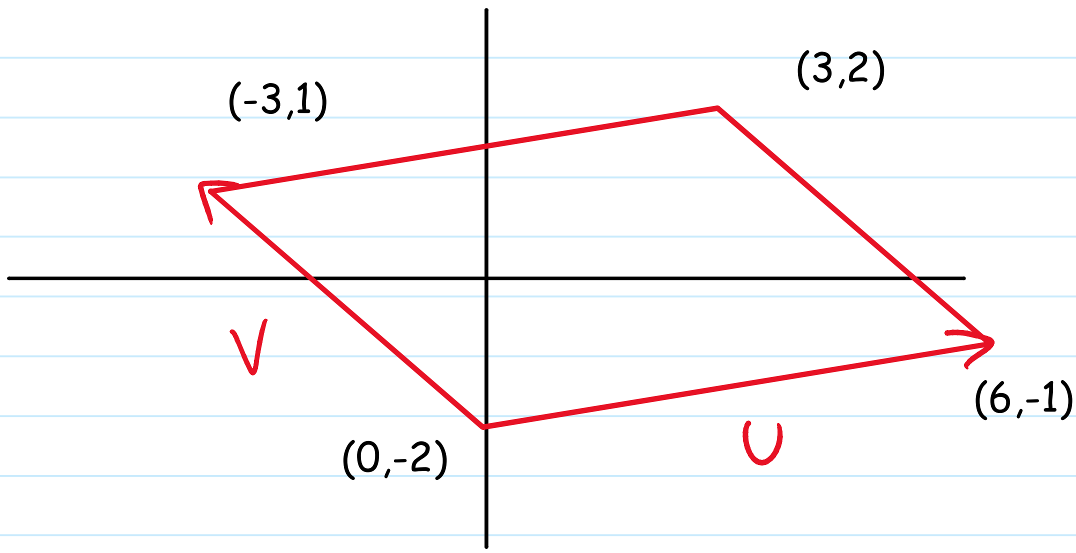
(**THIS** is how you get the formula for 2x2 inverses)

III- AREAS OF PARALLELOGRAMS

The final three applications are more geometric in nature

Example: Find the area of the parallelogram with vertices
(0,-2), (6,-1), (-3,1), (3,2)

Picture:



$u = \begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$v = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

Fact: The area of the parallelogram determined by u and v is

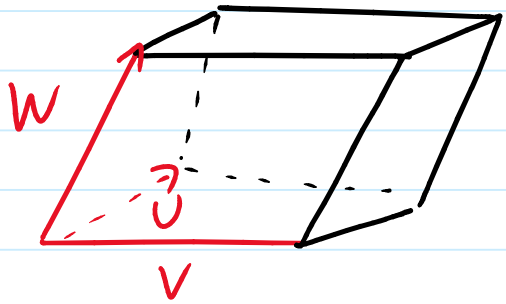
Area = | det[u | v] |

Here:

$$\left| \det \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix} \right| = |21| = 21$$

Similarly: Volume of Parallelipiped determined by u, v, w is

$$\text{Volume} = |\det[\, u \mid v \mid w \,]|$$



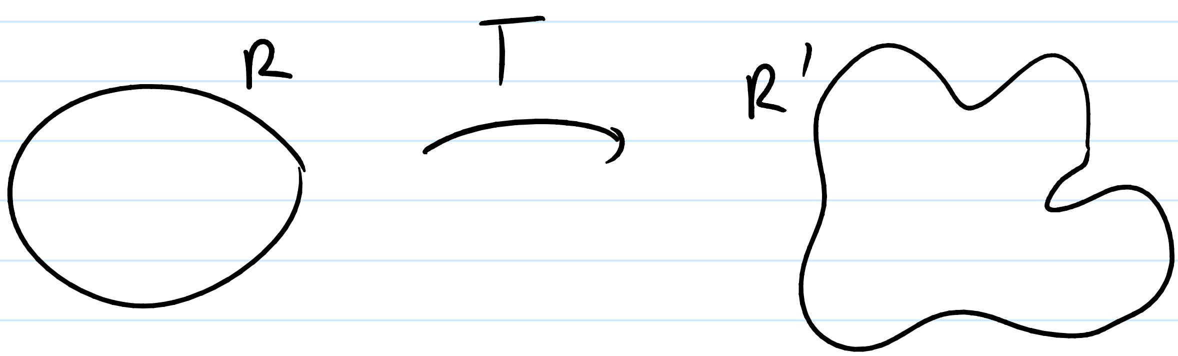
(Note: It doesn't matter if you put u and v as rows or as columns, since $\det(A^T) = \det(A)$)

IV- GENERAL AREAS

In fact, using determinants, you can also calculate areas and volumes of more general objects!

Setting:

Suppose you have a region R and apply a linear transformation T to get another region R'



Then:

FACT: $\text{Area}(R') = |\det(A)|\ \text{Area}(R)$ (*)

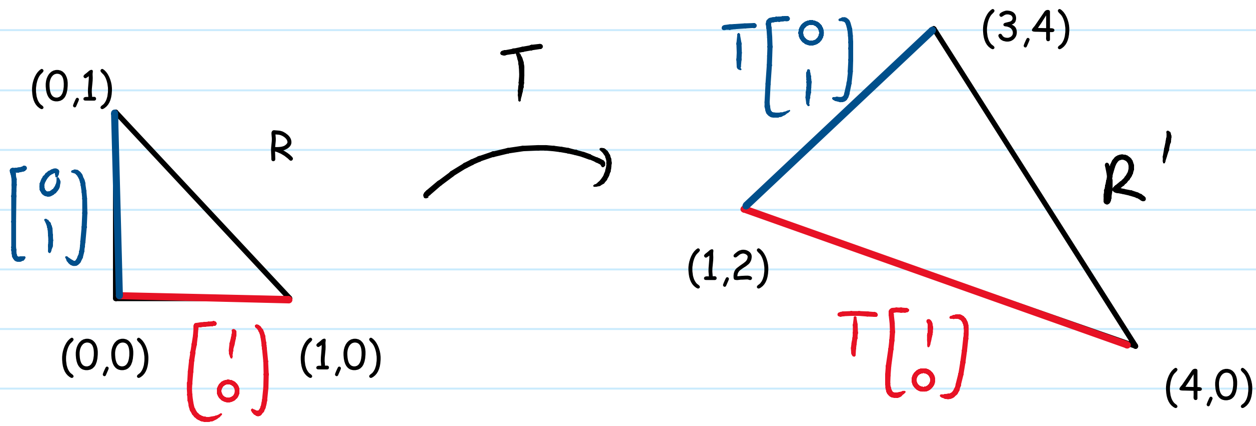
(A = matrix of T)

So the determinant is really a measure of how the area of R changes!

(**Optional Sidenote**: This is **PRECISELY** why in Math 2E, you have a determinant in the Jacobian; need to know how the area of the square dxdy changes when you change variables)

Example: (Similar to last problem on Winter 2019 exam)

Find the area of the triangle with vertices (1,2), (3,4), (4,0)



(Clever way: It's half a parallelogram! But here let's see *why* the formula for a parallelogram works)

STEP 1: Find R

Idea: Triangle = R'

Want to find R

Here: R = Triangle with vertices (0,0), (1,0), (0,1)
(always an easier version of R')

STEP 2: Find T (or A)

UPSHOT: Don't need to find T($\mathbf{x}$) for **every x** !!! Enough to find

$$T\begin{bmatrix}1\\0\end{bmatrix} \text{ and } T\begin{bmatrix}0\\1\end{bmatrix}$$

But

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$$

Hence:

$$A = \begin{bmatrix}3 & 2\\-2 & 2\end{bmatrix}$$

STEP 3: By (*),

Area(R') = |det(A)| Area(R)

$$= \left|\det\begin{bmatrix}2 & 2\\3 & -2\end{bmatrix}\right| (1/2 * 1 * 1) \quad \text{(R = right triangle w/ sides 1 \& 1)}$$

$$= |-10|\ (1/2)$$

$= 10/2$
$= 5$

(**Optional Sidenote**: Technically, here T not a linear transformation since $T(0,0) \neq (0,0)$; so in theory, need to shift the triangle so that the vertex (1,2) becomes (0,0) and re-do this problem. But the same method works here anyway…)

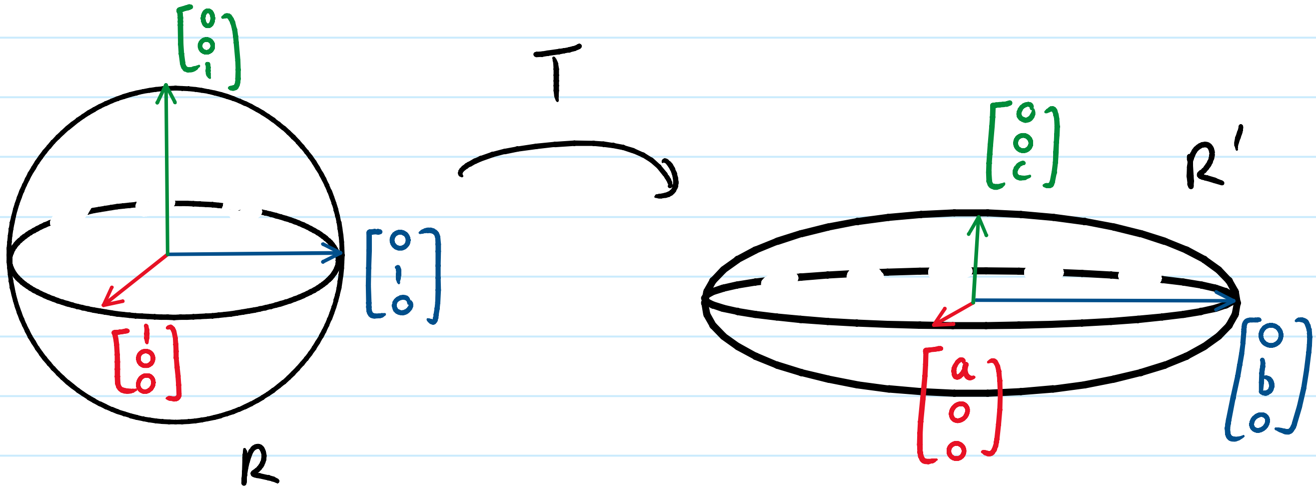
V- AS EASY AS 4/3 π abc

In fact, the **exact** same formula not only holds for areas, but also for volumes!

$$\boxed{\text{Fact: } \mathrm{Vol}(R') = |\det(A)|\, \mathrm{Vol}(R)}$$

Example: Find the volume of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$

(Compare with Math 2D/2E; here no messy triple integrals, just some clean linear algebra!)



STEP 1: Here

R'= ellipsoid

R = unit ball, $x^2 + y^2 + z^2 \le 1$

(**Why?** Ellipsoid is a stretched version of unit ball)

STEP 2: Find A

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a\\0\\0\end{bmatrix} \qquad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\b\\0\end{bmatrix} \qquad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\c\end{bmatrix}$$

$$A = \begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix}$$

STEP 3:

$$Vol(R') = |det(A)|\ Vol(R)$$

$$= \left| \det \begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix} \right| (4/3)\,\pi\,(1)^3 \quad (R = \text{unit ball})$$

$$= |abc|\ 4/3\ \pi$$

$$= 4/3\ \pi\ abc$$

WOW, no weird integration required!