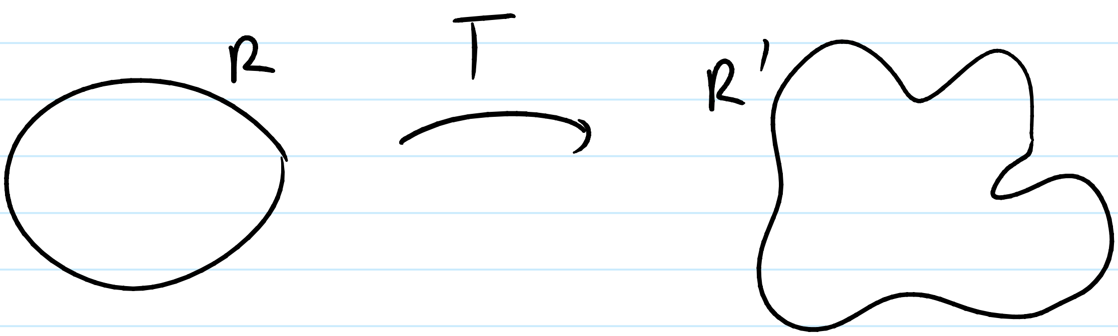
LECTURE 15: MIDTERM REVIEW

Monday, October 28, 2019 12:44 PM

Today: All about True/False Questions, but first let me wrap up one more thing from 3.3 (which is fair game for the exam)

I- AS EASY AS 4/3 π abc

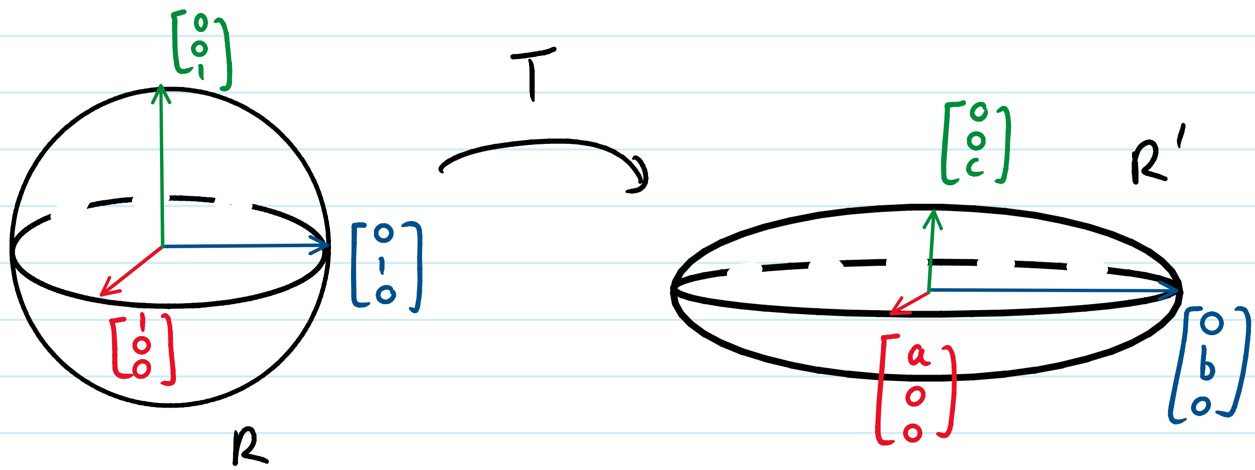
Recall: If R is a region and you apply a linear transformation T to it to get another region R'



$\boxed{\textbf{Fact}\text{: Vol}(R') = |\det(A)|\ \text{Vol}(R)}$

Example: Find the volume of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$

(Compare with Math 2D/2E; here no messy triple integrals, just some clean linear algebra!)



STEP 1: Here

R' = ellipsoid
R = unit ball, $x^2 + y^2 + z^2 \leq 1$

(**Why?** Ellipsoid is a stretched version of unit ball)

STEP 2: Find *A*

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a\\0\\0\end{bmatrix} \qquad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\b\\0\end{bmatrix} \qquad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\c\end{bmatrix}$$

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

STEP 3:

$$\mathrm{Vol}(R') = |\det(A)|\ \mathrm{Vol}(R)$$

$$= \left| \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right| (4/3)\,\pi\,(1)^3 \quad (R = \text{unit ball})$$

$$= |abc|\ 4/3\,\pi$$

$$= 4/3\,\pi\, abc$$

WOW, no weird integration required!

II- TRUE/FALSE EXTRAVAGANZA!

Remember: 111 T/F questions are available on YouTube!

F (a) R^2 is a subspace of R^3

R^2 **CANNOT** be a subspace of R^3 because R^2 is not even a subSET of R^3

How can $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be of the same form as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

(If you plug in $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in a program that accepts only things like $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
You'll get an error message)

BUT $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle|\ x, y \text{ in } R \right\}$ **IS** a subspace of R^3

F (b) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle|\ y \geq 0 \right\}$ is a subspace of R^2

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in H but $(-2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ is not in H $(-2 < 0)$

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T (c) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x^2 + y^2 = 0 \right\}$ is a subspace of R^2

You MIGHT think that it's not because there are squares, but

$H = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ which IS a subspace of R^2

(Important to find explicit counterexample: If you try it out, you'll see there is none)

T (d) If A (m x n) has n pivot columns, then Nul(A) = {0}

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix} \end{array}$$

Pivot in every column => No free variables

=> $A\mathbf{x} = \mathbf{0}$ => $\mathbf{x} = \mathbf{0}$

=> Nul(A) = {0}

F (e) $$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{vmatrix} = 123$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \boxed{0 \;\; -1 \;\; -2 \;\; -3 \;\; -4} \\ \boxed{0 \;\; -2 \;\; -4 \;\; -6 \;\; -8} \\ \cdots \end{vmatrix}$$

LIN DEP!

$$\begin{vmatrix} \vdots \\ \boxed{0 \ \ -2 \ \ -4 \ \ -6 \ \ -8} \\ \vdots \end{vmatrix}$$ LIN DEP!

$= 0$ (not invertible)

T (f) The matrix $AB + B^T A^T$ is always symmetric ($A^T = A$)

$(AB + B^T A^T)^T$
$= (AB)^T + (B^T A^T)^T$
$= B^T A^T + (A^T)^T (B^T)^T$
$= B^T A^T + AB$
$= AB + B^T A^T$

F (g) If $Ax = b$ is consistent for every b, then $Ax = 0$ only has the trivial solution

WARNING: For IMT, unless I say A is square, do NOT assume A is square

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$Ax = b$ is consistent for every b (pivot in every row), but $Ax = 0$ has a nonzero solution (because 1 free variable)

T (h) Same, but A is $n \times n$

By IMT, A is invertible

F (i) If $A \sim B$ ($\sim$ = row-equivalent), then $\mathrm{Col}(A) = \mathrm{Col}(B)$

$$\begin{array}{ccc} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \sim & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ A & & B \end{array}$$

But Col(A) = Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Col(B) = Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

THIS is why it's important to go back to the original pivot columns!

T (j) If A is a 4 x 9 matrix, then Nul(A) is at least 5 dimensional

$$A = \begin{array}{cc} & 9 \\ 4 & \begin{bmatrix} * & & & & & & & & \\ & * & & & & & & & \\ & & * & & & & & & \\ & & & * & & & & & \end{bmatrix} \end{array}$$

Rank(A) + dim(Nul(A)) = 9
=> dim(Nul(A)) = 9 - Rank(A)

But rank(A) 4 (because at most 4 pivots)
=> dim(Nul(A)) 9 - 4 = 5

F (k) If A is a 4 x 6 matrix with 2 pivot columns, then Nul(A) = R^4

CANNOT be, Nul(A) is a subspace of R^6 !

T (l) Every linear transformation T: R^n -> R^m has a matrix

$A = [T(e_1)\ T(e_2) \dots T(e_n)]$

T (m) If $\det(A^2) - 2\det(A) + \det(I) = 0$, then A is invertible

$(\det(A))^2 - 2\det(A) + 1 = 1\ 0$
$(\det(A) - 1)^2 = 0$
$\mathrm{Det}(A) - 1 = 0$

Det(A) = 1 0, so yet

F (n) The following set is a basis for R^3

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

$\times -2$, $\times -3$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -18 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Only 2 pivots, so NO

F (o) If AB = I, then BA = I

AGAIN, do NOT assume A is square!

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Can check AB = I but BA = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

F (p) $A^2 = O \Rightarrow A = O$ (O = zero matrix)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Point: Matrix Algebra is weird, not like usual algebra!

F (q) A system with 2 equations in 3 unknowns always has a solution

F (q) A system with 2 equations in 3 unknowns always has a solution

$$\begin{cases} x + y + z = 2 \\ x + y + z = 3 \end{cases}$$

F (r) If {u, v, w} is linearly dependent, then u is a linear combo of v and w

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$