Last time: We learned about matrix operations and how to find the inverse of a matrix:

**Definition:** $A^{-1}$ is the matrix such that:

$$A^{-1} A = A A^{-1} = I$$

Today:

1) How to find $A^{-1}$?
2) *When* can we find $A^{-1}$? *When* is $A$ invertible?

**I-HOW TO FIND $A^{-1}$**

**Example:** Find $A^{-1}$ where:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

**STEP 1** Form a HUGE matrix:

$$[A \mid I] = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

**STEP 2** Row reduce until you get

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \text{BLAH} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
(basically RREF, don't overthink it)

\[
\begin{bmatrix}
A & \mathbf{1}
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 2 & 3 & 0 & 0 & 1
\end{bmatrix}
\]

RREF

\[
\begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & -2 & 2 & 1
\end{bmatrix}
\]

(not enough!)

RREF

\[
\begin{bmatrix}
1 & 0 & 0 & 2 & -1 & -1 \\
0 & 1 & 0 & 3 & -1 & -1 \\
0 & 0 & 1 & -2 & 2 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{1} & A^{-1}
\end{bmatrix}
\]

Answer:

\[
A^{-1} = \begin{bmatrix}
2 & -1 & -1 \\
3 & -1 & -1 \\
-2 & 2 & 1
\end{bmatrix}
\]

Example: Find $A^{-1}$ where

\[
A = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\]

$(\times \mathbb{I})$\quad \[
\begin{bmatrix}
\mathbf{1} & -1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}
\]
II- ELEMENTARY MATRICES

You might ask: "Why in the world does this technique work?"

For this, we need to learn just a little bit about elementary matrices

**FACT 1: Can write EROS in terms of "elementary" matrices**

**Type 1: Multiply a row**

\[
\begin{bmatrix}
1 & -1 & 1 \\
0 & 2 & -1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
2 & 0 & 3 \\
-1 & 1 & 0
\end{bmatrix}
\div 2
\]

\[
\rightarrow \begin{bmatrix}
1 & -1 & 1 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 2
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
0 & 1 & \frac{1}{2} \\
0 & 1 & \frac{1}{2} \\
1 & 2 & \frac{1}{2}
\end{bmatrix}
\]

\[
A^{-1} = \begin{bmatrix}
\frac{1}{4} & \frac{1}{2} \\
-\frac{1}{4} & \frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 15 & 18 \\
0 & 0 & 1
\end{bmatrix}
\]

Multiplies second Row by 3
Type 2: Interchange two rows

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
= 
\begin{bmatrix}
7 & 8 & 9 \\
4 & 5 & 6 \\
1 & 2 & 3 \\
\end{bmatrix}
\]

Interchanges
Rows 1 & 3

(like the identity matrix I, but rows 1 and 3 are swapped)

Type 3: Add a row to another

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
11 & 16 & 21 \\
\end{bmatrix}
\]

Adds 4 times the 1st row
To the 3rd row

(like I but (3,1)st entry is 4 instead of 0)

FACT 2: Row-reducing is like multiplying by a big matrix R (= product of elementary matrices)

Now let me explain why the above procedure works
In terms of matrices, this means:

So \( RA = I \) and \( R = ? \)

But \( RA = I \) (and \( A \) is square) \( \Rightarrow R = A^{-1} \)

But also \( R = ? \), so \( ? = A^{-1} \)

And this is why we always get \( A^{-1} \) on the right!

**III- INTERPRETATION OF \( A^{-1} \)**

Just like we were able to give a linear transformation interpretation of \( AB \) (in terms of composition), we can also give a LT interpretation of \( A^{-1} \).
If $T$ is a LT, then $T^{-1}$ (inverse transformation) is defined by:

$$T(x) = y \iff T^{-1}(y) = x$$

**Interpretation:** If $T$ is a slight, $T^{-1}$ is the return flight. Whenever $T$ brings you somewhere, $T^{-1}$ brings you back.

In particular $T^{-1}(T(x)) = x$

**Fact:** If the matrix of $T$ is $A$, then the matrix of $T^{-1}$ is $A^{-1}$

(that’s why it’s called the inverse of $A$)

**Consequences:**

1) $(AB)^{-1} = B^{-1} A^{-1}$
   
   (reverse order! If you put your socks on and then your shoes, you first remove your shoes and then your socks)

2) $(A^{-1})^{-1} = A$

**IV- INVERTIBILITY (section 2.3)**

**Question:** Can we always find $A^{-1}$? Sadly no!
**Definition:** A is invertible if there is a matrix $B$ such that

$$AB = BA = I$$

**Example:** Is $A$ invertible?

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

For any $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

So $AB$ can never be $I$!

**Example:** "Find" $A^{-1}$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & I \end{pmatrix}$$

$$= (x^{-1}) \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$= (x^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$(x^{-1})$
In fact, $A$ is not invertible, you CANNOT find $B$ such that $AB = BA = I$!

Notice: Here $A$ only has 2 pivots!

And in fact, this leads us to:

**IV- THE INVERTIBLE MATRIX THEOREM (IMT)**

Tells us:

1) *When* a matrix is invertible
2) Invertible matrices are nice

Long Theorem, **BUT** it’s just the Row Theorem and the Column Theorem combined

Keep the following example in mind for the following theorem:
INVERTIBLE MATRIX THEOREM (IMT):

Let $A$ be $n \times n$ (!!!), then the following are equivalent

1) $A$ is invertible ($AB = BA = I$)
2) $A$ has $n$ pivots
3) $Ax = b$ is consistent for every $b$
4) Span of Columns of $A$ is $\mathbb{R}^n$ (remember $m = n$)
5) $T(x) = Ax$ is onto $\mathbb{R}^n$
6) $Ax = 0 \Rightarrow x = 0$
7) Columns of $A$ are LI
8) $T(x) = Ax$ is one to one
9) $(BA = I$ for some $B$)
10) $(AB = I$ for some $B$)
11) $(Ax = b$ has exactly one solution)

(Examples next time)