LECTURE 9: THE INVERSE OF A MATRIX

Monday, October 14, 2019 12:28 PM

Last time: We learned about matrix operations and how to find the inverse of a matrix:

Definition: A^{-1} is the matrix such that:

$$\boxed{A^{-1} A = A A^{-1} = I}$$

Today:

1) How to find A^{-1} ?
2) When can we find A^{-1} ? When is A *invertible*?

I- HOW TO FIND A^{-1}

Example: Find A^{-1} where:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

STEP 1 Form a HUGE matrix:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

STEP 2 Row reduce until you get

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \text{BLAH} \\ 0 & 0 & 1 & \end{array}\right]$$

(basically RREF, don't overthink it)

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{REF}} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

(not enough!)

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

$$[I \mid A^{-1}]$$

Answer: $$\boxed{A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 3 & -1 & -1 \\ -2 & 2 & 1 \end{bmatrix}}$$

Example: Find A^{-1} where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$(\times -1) \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] (\div 2)$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] (\times 1)$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}}$$

II- ELEMENTARY MATRICES

You might ask: "Why in the world does this technique work?"

For this, we need to learn just a little bit about elementary matrices

FACT 1: Can write EROS in terms of "elementary" matrices

Type 1: Multiply a row

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Multiplies second Row by 3}} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 15 & 18 \\ 0 & 0 & 1 \end{bmatrix}$$

Type 2: Interchange two rows

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Interchanges
Rows 1 & 3

(like the identity matrix I, but rows 1 and 3 are swapped)

Type 3: Add a row to another

$$\begin{array}{c c} & \begin{array}{ccc} 1 & & \\ \downarrow & & \end{array} \\ 3 \leftarrow & \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}} \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 11 & 16 & 21 \end{bmatrix}$$

Adds 4 times the 1st row
To the 3rd row

(like I but $(3,1)^{st}$ entry is 4 instead of 0)

FACT 2: Row-reducing is like multiplying by a big matrix R (= product of elementary matrices)

Now let me explain why the above procedure works

$$[A \mid I] \to [I \mid ?]$$

In terms of matrices, this means:

FACT 2

$$R[A \mid I] = [I \mid ?]$$

$$[RA \mid RI] = [I \mid ?]$$

$$[RA \mid R] = [I \mid ?]$$

So RA = I and R = ?

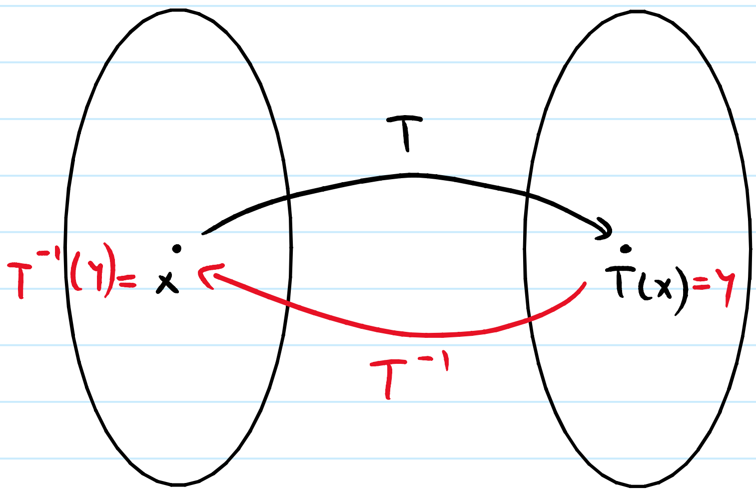
But RA = I (and A is square) => R = A^{-1}

But also R = ?, so ? = A^{-1}

And this is why we always get A^{-1} on the right!

III- INTERPRETATION OF A^{-1}

Just like we were able to give a linear transformation interpretation of AB (in terms of composition), we can also give a LT interpretation of A^{-1}



If T is a LT, then T^{-1} (inverse transformation) is defined by:

$$T(x) = y \iff T^{-1}(y) = x$$

Interpretation: If T is a slight, T^{-1} is the return flight. Whenever T brings you somewhere, T^{-1} brings you back.

In particular $T^{-1}(T(x)) = x$

Fact: If the matrix of T is A, then the matrix of T^{-1} is A^{-1}

(that's why it's called the inverse of A)

Consequences:

1) $(AB)^{-1} = B^{-1}A^{-1}$
   (reverse order! If you put your socks on and then your shoes, you first remove your shoes and then your socks)
2) $(A^{-1})^{-1} = A$

IV- INVERTIBILITY (section 2.3)

Question: Can we always find A^{-1} ? Sadly no!

Definition: A is invertible if there is a matrix B such that

$$AB = BA = I$$

Example: Is A invertible?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

For any $\quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So AB can never be I!

Example: "Find" A^{-1} where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[A \mid I]$$

$$= (\times -1)\left(\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \right.$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] (\times -1)$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

CANNOT turn this into [I | BLAH]

In fact, A is **not** invertible, you **CANNOT** find B such that AB = BA = I !

Notice: Here A only has 2 pivots!

And in fact, this leads us to:

IV- THE INVERTIBLE MATRIX THEOREM (IMT)

Tells us:

1) **When** a matrix is invertible
2) Invertible matrices are nice

Long Theorem, **BUT** it's just the Row Theorem and the Column Theorem combined

ROW
$Ax = b$
Span
Onto

$\xleftrightarrow{\text{IMT}}$

COLUMN
$Ax = 0$
LI
1 1

Keep the following example in mind for the following theorem:

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

INVERTIBLE MATRIX THEOREM (IMT):

Let A be $n \times n$ (!!!), then the following are equivalent

1) A is invertible ($AB = BA = I$)
2) A has n pivots
3) $Ax = b$ is consistent for every b
4) Span of Columns of A is R^n (remember $m = n$)
5) $T(x) = Ax$ is onto R^n
6) $Ax = 0 \Rightarrow x = 0$
7) Columns of A are LI
8) $T(x) = Ax$ is one to one
9) ($BA = I$ for some B)
10) ($AB = I$ for some B)
11) ($Ax = b$ has exactly one solution)

ROW THEOREM (3–5)

COLUMN THEOREM (6–8)

(Examples next time)